Decentralized College Admissions

Yeon-Koo Che and Youngwoo Koh*

October 19, 2013

Abstract

We develop a model of decentralized college admissions in which students' preferences for colleges are uncertain, and colleges incur costs whenever their enrollments exceed their capacities. Colleges' admission decisions become a tool for strategic yield management, because the enrollment at a college depends on not only students' uncertain preferences but also other colleges' admission decisions. We find that equilibrium admission decisions exhibit "strategic targeting"—colleges may forgo admitting (even good) students likely sought after by the others and may admit (not as good) students likely overlooked by the others. Randomization in admissions may also emerge. The resulting assignment is inefficient and leads to justified envy. When the colleges consider multiple dimensions of students merits, they may avoid head-on competition by placing excessive weights on less correlated dimensions, such as extracurricular activities and non-academic aspects. Restricting the number of applications or allowing for wait-listing might alleviate colleges' yield management problem, but the resulting assignments are still inefficient and admit justified envy. Centralized matching via Gale and Shapley's Deferred Acceptance algorithm eliminates the yield management problem and justified envy and attains efficiency. But, some colleges may be worse off relative to decentralized matching.

1 Introduction

The standard market design research on matching focuses on how best to design a centralized matching mechanism, taking the societal consensus on centralization as a given. While such a consensus exists in a number of markets (e.g., medical residency matching and public school matching), many markets remain decentralized (e.g., college admissions and graduate school admissions). Decentralized markets often exhibit congestion and do not operate efficiently (Roth and Xing, 1997). Although it is widely believed that these markets will benefit from improved coordination or centralization, it is not well understood why they remain decentralized and what welfare benefits would be gained by improving coordination possibly via a centralized clearinghouse.

*Che: Department of Economics, Columbia University, yeonkooche@gmail.com; Koh: College of Economics and Finance, Hanyang University, youngwoo.koh@gmail.com

At least part of the problem is the lack of an analytical grasp of decentralized matching markets. Often treated as a black box, the equilibrium and welfare implications of decentralized matching markets have not been understood well in the literature. Indeed, we have yet to develop a workhorse model of decentralized matching that could serve as a useful benchmark for comparison with a centralized system.[1]

The current paper develops an analytical framework for understanding decentralized matching markets in the context of college admissions. In many countries, such as Japan, Korea, and the US, college admissions are organized similarly to decentralized labor markets, with exploding and binding admissions made by schools during a short window of time, among other things.

With limited offers and acceptances to clear the markets, decentralized matching provides only a limited chance for colleges to learn students' preferences and to condition their admission decisions on them. This presents a challenge for colleges in managing its yield. Inability to forecast yield accurately could result in too many or too few students enrolling in a college relative to its capacity. Either mistake is costly. For instance, 1,415 freshmen accepted Yale's invitation to join its incoming class in 1995-96, although the university had aimed for a class of 1,335. At the same year, Princeton also reported 1,100 entering students, the largest in its history. The college set up mobile homes in fields and built new dorms to accommodate the students (Avery, Fairbanks and Zeckhauser, 2003).[2]

The yield management problem becomes increasingly important in many countries. In Korea, for example, students apply for departments and not for colleges. Since each department has a small quota and there are many potential choices for students, it is critical for departments to predict yield rates accurately to ensure that they fill their capacities. In the US, most colleges continue to experience increase in the number of applications they receive,[3] and the average yield rate of four-year colleges in the US has declined significantly over the past decade, from 49 percent in 2001 to 38 percent in 2011 (Clinedinst, Hurley and Hawkins, 2012). Declining rates signal greatly increased uncertainty for colleges:

> *Trying to hit those numbers is like trying to hit hot tub when you are skydiving 30,000 feet. I'm going to go to church every day in April.* – Jennifer Delahunty (Dean

[1]The main exceptions are two excellent works by Chade and Smith (2006) and Chade, Lewis and Smith (2011). As we discuss more fully later, they focus on the portfolio decisions students face in application and colleges' inference of students' abilities based on imperfect signals. By contrast, the current paper focuses on the matching implications of college admissions, paying special attention to the yield management problem arising from (aggregately) uncertain students' preferences.

[2]The cost may also take the form of an explicit sanction imposed on the admitting unit (e.g., department) by the government (as in Korea) or by the college (as in Australia).

[3]The application increase in recent years is due partly to the increased number of high school graduates but mainly to an increase in applications submitted per student as online applications become prevalent. Seventy-nine percent of Fall 2011 freshmen applied to three or more colleges and twenty-nine percent of them submitted seven or more applications. (Clinedinst, Hurley and Hawkins, 2012)

of admissions and financial aid at Kenyon College in Ohio)[4]

Importantly, the uncertainty facing a college with respect to a student's enrollment depends not just on her preference but also on what other set of admissions she receives. This makes a college's admission policy a strategic yield management decision. We provide a simple model of colleges' strategic yield management problems and characterize the equilibrium outcomes of these strategic decisions. The explicit analysis of equilibrium allows us to evaluate the resulting assignment in terms of welfare and fairness and to compare this with outcomes that arise from other coordinated admissions and centralized matching.

In our baseline model, there are two colleges, each with limited capacity, and a unit mass of students with "scores" that are common for both colleges (e.g., high school GPA or SAT scores). Students apply to colleges at no cost. Colleges prefer students according to their scores, but they do not know students' preferences toward them. This uncertainty takes an aggregate form: The mass of students preferring one college over the other varies across states that are unknown to the colleges. Over-enrollment costs a college in proportion to the enrollment in excess of its capacity. Our baseline model involves a simple time line: Initially, students simultaneously apply to colleges. Each college observes only the scores of those students who apply to it. Next, the two colleges simultaneously offer admissions to sets of students. Finally, the students who are admitted by either or both colleges decide on which admission they will accept.

Given that application is costless, students have a (weak) dominant strategy of applying to both colleges. Hence, the main focus of the analysis is the college's admission decisions. Our main finding in this regard is that the colleges engage in "strategic targeting": In equilibrium, each college may forgo good students who are sought after by the other college and may admit less attractive students who appear overlooked by the other college. The reason for this is that the students who attract competing admissions from the other college present greater enrollment uncertainty and add to capacity cost. Randomization in admissions for students may also emerge. We then provide existence of these equilibria. Next, we show that the assignment is typically unfair; that is, it entails justified envy among students and fails to achieve efficiency among students, among colleges and among all parties including colleges and students.

These results can be illustrated via a simple example. Suppose there are only two students, 1 and 2, applying to colleges A and B. Each college has one seat to fill and faces a prohibitively high cost of having two students. Student i has score v_i, $i = 1, 2$, where $0 < v_2 < v_1 < 2v_2$. Each student has an equal probability of preferring either school, which is private information (unknown to the other student and to the colleges). Each college values having student i at v_i. The applications are free of cost, and the timing is the same as that explained above.

Given the large cost of over-enrollment, each college admits only one of the students. Their

[4] "In Shifting Era of Admissions, Colleges Sweat," *NY Times*, March 8, 2009

payoffs are described as follow.

A's strategy\ B's strategy	Admit 1	Admit 2
Admit 1	$\frac{1}{2}v_1,\ \frac{1}{2}v_1$	v_1, v_2
Admit 2	v_2, v_1	$\frac{1}{2}v_2,\ \frac{1}{2}v_2$

This game has a battle of the sexes' structure (with asymmetric payoffs), so it is not difficult to see that there are two different types of equilibria. First, there are two asymmetric pure-strategy equilibria in which one college admits student 1 and the other admits student 2. There is also a mixed-strategy equilibrium in which each college admits 1 with probability $\gamma := \frac{2v_1 - v_2}{v_1 + v_2} > 1/2$ and admits 2 with probability $1 - \gamma$, where γ is chosen such that the other college is indifferent. Both types of equilibria show the pattern of strategic targeting. In the pure-strategy equilibria, colleges manage to avoid competition and thus randomness in enrollment by targeting different students. The mixed-strategy equilibrium can be interpreted as arising from strategic targeting, i.e., colleges' attempt to avoid students sought after by the other, although it does not result in perfect coordination.

This example, while extremely simple, suggests problems with decentralized matching in terms of welfare and fairness. First, the student with high score (student 1) may be assigned to a less preferred school (in both types of equilibria) even though both colleges prefer the high scoring student; that is, justified envy arises. Second, it could be the case that student 1 prefers A and student 2 prefers B, but the former is assigned to B and the latter is assigned to A, showing that the equilibrium outcome is inefficient among students. Lastly, the mixed-strategy equilibrium is Pareto inefficient because both colleges may admit the same student, in which case one college is unmatched and would rather match with the other student.

We next study the admissions problem when students have multidimensional types. Some measures, such as students' academic performances or system wide test like SAT, are highly correlated among colleges, but others measures, such as students' extracurricular activities or college specific tests and essays, are less correlated among them.[5] Clinedinst, Hurley and Hawkins (2012) report that private colleges place emphasis on many factors other than standard test scores, including essay/writing sample and extracurricular activities. We show that colleges' desire to avoid head-on competition, and thus to lessen enrollment uncertainty, leads them to bias their evaluation toward less correlated measures by placing excessive weights on theses dimensions.

We also study two common ways for colleges to alleviate their yield management problem. One common way is "self-targeting," whereby colleges coordinate to restrict the number of applications each student can submit. This form of coordination is observed in many countries; for instance,

[5]In Korea, for instance, students take a nationwide exam and each college has its own essay tests and/or oral interviews. In Japan, there is a nationwide exam called National Center Test (NCT). Public universities use both NCT and their own exams, and private universities often use their own exam only.

students in the UK cannot apply to both Cambridge and Oxford, students in Japan can apply to at most one public university, and students in Korea face a similar restriction. Self-targeting reduces the enrollment uncertainty for the colleges, and thus alleviates their yield management burden. Yet, we show that this method may not completely eliminate the yield management problem and justified envy, and it may also fail to achieve efficiency.

Another way to cope with the enrollment uncertainty is by admitting students in sequence, or "wait-listing": Colleges admit some students and place others in the wait list in each of multiple rounds and later extend admissions to those in the wait list when seats open up from the previous round. This method is also observed in many countries, including France, Korea and the US. Wait-listing alleviates colleges' yield management problem, since colleges may adjust their admission offers based on the students' acceptance behavior and the information the colleges may learn over the course of the process. We show, however, that they still engage in strategic targeting under this mechanism, and the welfare and fairness problems still remain.

Finally, we consider a centralized matching via Gale and Shapley's Deferred Acceptance algorithm (DA in short). This eliminates colleges' yield management problem and justified envy completely and attains efficiency. At the same time, it is possible for one college to be worse off relative to the decentralized matching. For instance, in the above example, suppose a pure-strategy equilibrium in which college i always gets student 1 is played. Then, that college will clearly be worse off from a switch to a centralization via DA because it will not always attract student 1. This may explain a possible lack of consensus toward centralization and may underscore why college admissions remain decentralized in many countries.

The paper is organized as follows. Section 1.1 discusses the related literature. The model is introduced in Section 2. Equilibrium is characterized in Section 3. Section 3.1 establishes existence of equilibrium. Section 3.2 discusses welfare and fairness implications of equilibria. Section 4 studies admissions problem when students' types are multidimensional. In Section 5, self-targeting via restriction on the number of applications is studied, and in Section 6, wait-listing is studied. Centralized matching via DA is considered in Section 7. Section 8 concludes the paper. Proofs are provided in the Appendix unless stated otherwise. The Appendix also extends the baseline model to allow for more than two colleges and shows that our analysis in the two-college model carries over.

1.1 Related Literature

Several papers in the matching literature have considered decentralized matching markets. Roth and Xing (1997) study the entry-level market for clinical psychologists in which firms make offers to workers sequentially within a day and workers can accept, reject or hold an offer. They find that, mainly based on simulations, such a decentralized (but coordinated) market exhibits *congestion*, i.e., not enough offers and acceptances could be made to clear the market, and the resulting outcome is unstable. Neiderle and Yariv (2009) also study a decentralized (one-to-one matching) market in which firms make offers sequentially through multiple periods. They provide sufficient conditions under which such decentralized markets generate stable outcomes in equilibrium in the presence of market friction (namely, time discounting) and preference uncertainty. Like these models, our model concerns the consequence of congestion arising from decentralized matching, but unlike Roth and Xing (1997), we study participants' strategic responses, analyzing equilibrium admission decisions and their welfare and fairness properties. In particular, the current framework develops a new theme of strategic targeting. Moreover, the explicit analysis of equilibria permits a clear comparison with the outcome that would arise from a centralized matching.

The college admissions problem has recently received attention in the economics literature. Chade and Smith (2006) study students' application decision as a portfolio choice problem. Chade, Lewis and Smith (2011) analyze colleges' admission decisions together with the students' application decisions. In their model, students with heterogeneous abilities make application decisions subject to application costs, and colleges set admission standards based on noisy signals on students' abilities. Avery and Levin (2010) and Lee (2009) study early admissions. Unlike our model, these models have no aggregate uncertainty with respect to students' preferences, which means that the colleges in their models do not face any enrollment uncertainty. Hence, colleges do not employ strategic targeting; they instead use cutoff strategies.

Some aspects of our equilibrium are related to political lobbying behavior studied by Lizzeri and Persico (2001, 2005). Just as colleges target students in our model, politicians in these models target voters for distributing their favors. In their models, voters are homogeneous, and a voter votes for the candidate that offers her the largest favor. In our model, however, students have heterogeneous abilities and preferences. Thus, colleges' admission decisions are more complicated—admission probabilities vary according to students' scores. Aggregate uncertainty plays a unique role in shaping competition in our model, whereas how the spoils of office are split among candidate (either *winner-take-all* or *proportional* rule) is crucial in their model.

Our model also shares some similarities with directed search models, such as Montgomery (1991) and Burdett, Shi and Wright (2001). In these studies, each firm (seller) posts a wage (price), and each worker (buyer) decides which job to apply for. Firms have a fixed number of job openings and cannot hire more than the capacity, and workers can only apply to one firm. Workers' inability to precisely coordinate their search decisions causes a "search friction," so they randomize on application decisions. Just like the workers in these models, colleges in our model can be seen to engage in "directed searches" on students. The difference is that the colleges in our model offer admissions to many students subject to aggregate uncertainty. This leads to strategic targeting, a novel feature of our model.

2 Model

There is a unit mass of students with score v distributed from $\mathcal{V} \equiv [0,1]$ according to an absolutely continuous distribution $G(\cdot)$. There are two colleges, A and B, each with capacity $\kappa < \frac{1}{2}$. (Appendix B will extend the model to include more than two colleges, showing that our main results carry over to that extension.) Each college values a student with score v at v and faces a cost $\lambda \geq 1$ for each incremental enrollment exceeding the quota. Each student has a preference over the two colleges, which is private information. A state of nature s, drawn from $[0,1]$ according to the uniform distribution, determines the fraction of students who prefer A over B. In state s, a fraction $\mu(s) \in [0,1]$ of students prefers A to B, where $\mu(\cdot)$ is strictly increasing and continuous in s.[6] While we shall consider a general environment with respect to $\mu(\cdot)$, some result will consider a **symmetric environment** in which $\mu(s) = 1 - \mu(1-s)$ for all $s \in [0,1]$. In a symmetric environment, the measure of students who prefer A over B is symmetric around $s = \frac{1}{2}$.

The timing of the game is as follows. First, Nature draws the (aggregate uncertainty) state s. Next, all students simultaneously apply to colleges. Each college observes the scores of only those students who apply to it. Next, colleges simultaneously decide which applicants to admit. Last, students who have received at least one admission offer decide which offer to accept.

We assume that there is no application cost for the students, so it is a weak dominant strategy for each student to apply to both colleges. Throughout this paper, we focus on a perfect Bayesian equilibrium in which students play the weak dominant strategy.[7]

Colleges distribute admissions based on students' scores. Let $\alpha : \mathcal{V} \to [0,1]$ and $\beta : \mathcal{V} \to [0,1]$ be college A and B's admission strategies, respectively, in terms of the fractions of students with score v colleges admit.

For given $\alpha(\cdot)$ and $\beta(\cdot)$, let $\mathcal{V}_A := \{v \in [0,1] \,|\, \alpha(v) > 0\}$ and $\mathcal{V}_B := \{v \in [0,1] \,|\, \beta(v) > 0\}$ be the types of students colleges A and B respectively admit. Let $\mathcal{V}_{AB} := \mathcal{V}_A \cap \mathcal{V}_B$. If $\mathcal{V}_{AB}$ has a positive measure in an equilibrium, this means that a positive measure of students has admissions from both colleges. We call such an equilibrium **competitive**. An equilibrium in which $\mathcal{V}_{AB}$ has zero measure is called **non-competitive**.

Consider the students with score v. A fraction $\alpha(v)[1 - b(v)]$ of them is admitted only by A, and a fraction $\alpha(v)\beta(v)$ of them is admitted by both colleges, in which case a fraction $\mu(s)$ of those latter students prefers A over B. Thus, the mass of students who attend A in state s, given

[6] There is no loss of generality to assume the uniform distribution, because for a distribution $F(\cdot)$ of s, we can simply relabel s, and the popularity of a college over the other is captured by $\mu(\cdot)$.

[7] The strategy of applying to both colleges can be made a strictly dominant strategy if students have some uncertainty about their scores, which is realistic in case the scores are either not publicly observable or depend on multiple dimensions of attributes, the weighting of which may be unknown to the students.

strategies $\alpha(\cdot)$ and $\beta(\cdot)$, is

$$m_A(s) := \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]\, dG(v). \tag{2.1}$$

Similarly, the mass of students who attend B in state s is

$$m_B(s) := \int_0^1 \beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)]\, dG(v). \tag{2.2}$$

Each college realizes the scores of enrolled students as its gross payoff and incurs cost λ for each increment beyond its capacity. Thus, college A and B's ex ante payoffs are, respectively,

$$\pi_A := \mathbb{E}_s\Big[\int_0^1 v\alpha(v)[1 - \beta(v) + \mu(s)\beta(v)]\, dG(v) - \lambda \max\{m_A(s) - \kappa, 0\}\Big]$$

and

$$\pi_B := \mathbb{E}_s\Big[\int_0^1 v\beta(v)[1 - \alpha(v) + (1 - \mu(s))\alpha(v)]\, dG(v) - \lambda \max\{m_B(s) - \kappa, 0\}\Big].$$

One immediate observation is that each college's payoff is concave in its own admission strategy (see Lemma A3), that is, $\pi_A(\eta\alpha + (1-\eta)\alpha') \geq \eta\, \pi_A(\alpha) + (1-\eta)\pi_A(\alpha')$ for any feasible strategies α and α' and for any $\eta \in [0,1]$. Therefore, mixing over α's is unprofitable for college A (similarly β's for college B). For this reason, any equilibrium is characterized by a pair (α, β). Of course, this does not mean that the equilibrium is in pure-strategies; the values of α and β may be strictly interior, in which case the admission strategies would involve randomization.

In the following sections, we characterize different types of equilibria and establish their existence. We then provide welfare and fairness properties of equilibria.

3 Characterization of Equilibrium

We analyze colleges' admission decisions in this section. To this end, we fix any equilibrium (α, β) and explore the properties it must satisfies. Later, we shall establish existence of the equilibria. We begin with the following observations, whose proofs are in Appendix A.1.

Lemma 1. *In any equilibrium (α, β), the following results hold.*

(i) $m_A(0) \leq \kappa \leq m_A(1)$ and $m_B(1) \leq \kappa \leq m_B(0)$.

(ii) $\mathcal{V}_A \cup \mathcal{V}_B$ is a connected interval with $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$ and $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$.

(iii) If the equilibrium is competitive (i.e., $\mathcal{V}_{AB}$ has a positive measure), then there exists a unique $(\hat{s}_A, \hat{s}_B) \in (0,1)^2$ such that $m_A(\hat{s}_A) = \kappa$ and $m_B(\hat{s}_B) = \kappa$.

(iv) If the equilibrium is non-competitive (i.e., $\mathcal{V}_{AB}$ has zero measure), then $m_A(s) = m_B(s) = \kappa$ for all $s \in [0,1]$. Further, almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives an admission offer from exactly one college.

Part (i) of the lemma states that in equilibrium, colleges cannot have strict over-enrollment and/or strict under-enrollment in all states. This is obvious since if there were over-enrollment in all states for a college, then since $\lambda \geq 1$, it will profitably deviate by rejecting some students with $v < 1$, and if there were under-enrollment in all states, a college will likewise profitably deviate by accepting more students. Part (ii) suggests that if a student with score v is admitted by either college, then all students with scores higher than such v must be admitted by some college at least with positive probability, and there is a positive mass of students in the low tail who are never admitted by either college. Part (iii) suggests that in a competitive equilibrium, the colleges will suffer from under-enrollment in some states and over-enrollment in other states. This is intuitive since given (aggregately) uncertain preferences on the part of students, the presence of students who receive admissions from both colleges presents non-trivial enrollment uncertainty. Each college will deal with uncertainty by optimally trading off the cost of over-enrollment with the loss from under-enrollment, thus entailing both types of mistakes depending on the states. Part (iv) states that in a non-competitive equilibrium, colleges avoid the over- and under-enrollment problems, and almost every top 2κ students receive admissions from only one college. This is, again, intuitive since the colleges in this case face no enrollment uncertainty, so they will fill their capacities exactly in all states with students whose scores are within the top 2κ.

In what follows, we shall focus on competitive equilibria. There are several reasons for this. It will be seen that competitive equilibria always exist (see Theorem 3). By contrast, non-competitive equilibria can be ruled out if either λ is not too large or κ is not too small (see Appendix A.2). Finally, even if a noncompetitive equilibrium exists, the characterization provided in Lemma 1-(iv) is sufficient for our welfare and fairness statements, as will be seen later.

Therefore, fix any competitive equilibrium (α, β). For ease of notation, let $\mu_+(s) := \mathbb{E}[\mu(\tilde{s})|\tilde{s} \geq s]$, $\mu_-(s) := \mathbb{E}[\mu(\tilde{s})|\tilde{s} \leq s]$ and $\overline{\mu} := \mathbb{E}[\mu(s)]$. It is convenient to rewrite A's payoff at the equilibrium as follows:

$$
\begin{aligned}
\pi_A &= \int_0^1 v\alpha(v)[1-\beta(v)+\overline{\mu}\beta(v)]\,dG(v) - \lambda\,\mathbb{E}[m_A(s)-\kappa\,|\,s > \hat{s}_A](1-\hat{s}_A) \\
&= \int_0^1 \alpha(v)\,H_A(v,\beta(v))\,dG(v) + \lambda\,(1-\hat{s}_A)\,\kappa,
\end{aligned}
$$

where $\hat{s}_A \in (0,1)$ is such that $m_A(\hat{s}_A) = \kappa$ as defined in Lemma 1-(iii), and

$$
\begin{aligned}
H_A(v,\beta(v)) &:= v[1-\beta(v)+\overline{\mu}\beta(v)] - \lambda(1-\hat{s}_A)[1-\beta(v)+\mu_+(\hat{s}_A)\beta(v)] \\
&= (1-\beta(v))\big[v-\lambda(1-\hat{s}_A)\big] + \beta(v)\overline{\mu}\Big[v-\lambda(1-\hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}\Big] \qquad (3.1)
\end{aligned}
$$

is A's marginal payoff from admitting a student with score v for given $\hat{s}_A$ and $\beta(\cdot)$[8] in equilibrium.

[8]We shall suppress its dependence on $\hat{s}_A$ unless it is important.

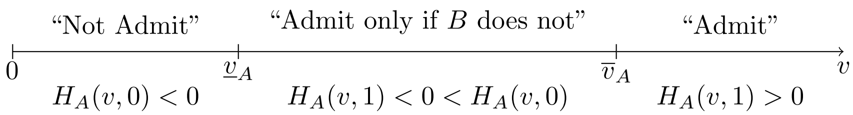


Figure 3.1: A's Admission Decision

H_A captures A's local incentive; that is, what A gains by admitting v, holding fixed its opponent's decision and *its own decisions for the rest of the students* at $\alpha(\cdot)$.

Notice that the first and the second square brackets in (3.1) are college A's marginal payoffs of admitting a student with score v when she does not receive an admission offer from B and when she does, respectively. Recall that the college incurs capacity cost only when there is over-enrollment. If the student does not receive a competing offer from B, then she accepts A's admission for sure. Hence, over-enrollment occurs with probability $(1-\hat{s}_A)$, entailing the marginal cost $\underline{v}_A := \lambda(1-\hat{s}_A)$, which explains the second term of the first square brackets in (3.1). If the student receives a competing offer from B, then she accepts A's offer only when she prefers A to B. Hence, conditional on acceptance, the over-enrollment arises with probability $(1-\hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}$, entailing the marginal cost $\overline{v}_A := \lambda(1-\hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}$, the second term in the second square brackets in (3.1).

Observe that $\mu_+(\hat{s}_A) > \overline{\mu}$ for $\hat{s}_A \in (0,1)$, so $\overline{v}_A$ is higher than $\underline{v}_A$. The reason for this is that when the student receives an offer from B but accepts A's offer, the state is more likely to be high comparing to the case that she does not receive a competing offer, since she is more likely to accept A's offer when $\mu(s)$ is high than when it is not. This explains why it is more costly to admit a student who is sought after by another college than a student who is not. Hence, college A is less likely to admit a student if college B admits her, and is more likely to admit the student if B does not.

Lemma 2. *In any competitive equilibrium, $H_i(v,x)$, $i = A, B$, is strictly increasing in v for each x. Moreover, for each v, $H_i(v,x)$ satisfies the single crossing property: If $H_i(v,x) \leq 0$ for some $x \in (0,1)$, then $H_i(v,x') < 0$ for any $x' > x$.*

Proof. See Appendix A.3. ■

Lemma 2 implies that $H_A(v,\beta(v))$ partitions the students' type space into three intervals, as depicted in Figure 3.1. First of all, there exist $\overline{v}_A > \underline{v}_A$ such that $H_A(\overline{v}_A,1) = 0$ and $H_A(\underline{v},0) = 0$. Since $H_A(v,1) > 0$ for $v > \overline{v}_A$ and $H_A(v,0) < 0$ for $v < \underline{v}_A$ (recall H_A is strictly increasing in v), college A admits all students with $v > \overline{v}_A$ *even if college B admits all of them* and rejects all students with $v < \underline{v}_A$ *even if college B rejects all of those students.*

For the students with $v \in (\underline{v}_A, \overline{v}_A)$, we have $H_A(v,1) < 0 < H_A(v,0)$. This means that college A's incentive for admitting these students depends on college B's admission decisions toward them. The single crossing property established in Lemma 2 implies that for each v, there exists $\hat{\sigma}_B(v) \in (0,1)$ such that $H_A(v,x) > 0$ if $x < \hat{\sigma}_B(v)$, $H_A(v,x) < 0$ if $x > \hat{\sigma}_B(v)$, and $H_A(v,\hat{\sigma}_B(v)) = 0$

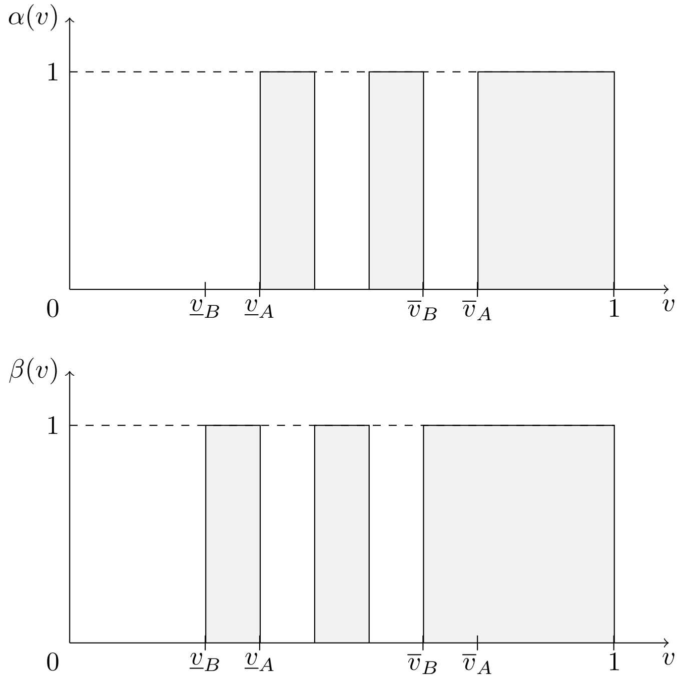


Figure 3.2: Pure-Strategy Equilibrium

if $x = \hat{\sigma}_B(v)$. Hence, college A admits all students with v if B admits (rejects) less (greater) than fraction $\hat{\sigma}_B(v)$ of them and admits any fraction of those students if B admits exactly fraction $\hat{\sigma}_B(v) \in (0,1)$ of them. In particular, college A admits all of them if B does not admit any of them, but does not admit them if B admits all of them.

The characterization of B's admission strategy is completely symmetric. Its payoff function is written as

$$\pi_B = \int_0^1 \beta(v) H_B(v, \alpha(v)) dG(v) + \lambda\, \hat{s}_B\, \kappa,$$

where

$$H_B(v, \alpha(v)) = (1 - \alpha(v))\big[v - \lambda \hat{s}_B\big] + \alpha(v)(1 - \overline{\mu})\Big[v - \lambda \hat{s}_B \frac{1 - \mu_-(\hat{s}_B)}{1 - \overline{\mu}}\Big],$$

and the admission strategy is described analogously using the cutoff scores $\underline{v}_B := \lambda\, \hat{s}_B$ and $\overline{v}_B := \lambda\, \hat{s}_B\, \frac{1-\mu_-(\hat{s}_B)}{1-\overline{\mu}}$, where $\underline{v}_B < \overline{v}_B$. Combining the two colleges' admission decisions leads to the following characterization of equilibria.

Theorem 1. *In any competitive equilibrium, there exist $\underline{v}_i < \overline{v}_i$, $i = A, B$, such that college i admits students with $v > \overline{v}_i$ and $v \in [\underline{v}_i, \underline{v}_j]$ and rejects students with $v < \underline{v}_i$ and $v \in [\overline{v}_j, \overline{v}_i]$, where $j \neq i$. At least one college admits a positive fraction of students with $v \in [\max\{\underline{v}_A, \underline{v}_B\}, \min\{\overline{v}_A, \overline{v}_B\}]$.*

Theorem 1 describes the structure of competitive equilibrium. Figure 3.2 depicts a typical pure-strategy equilibrium. Here, the students at the top with $v > \overline{v}_A = \max\{\overline{v}_A, \overline{v}_B\}$ receive admissions from both colleges, because their scores are above the high cutoffs of both colleges. And the students at the bottom below $\underline{v}_B = \min\{\underline{v}_A, \underline{v}_B\}$ do not receive any admissions. Strategic targeting occurs with students in the middle with $v \in [\underline{v}_B, \overline{v}_A]$. The students with $v \in [\overline{v}_B, \overline{v}_A]$ are admitted only by B, since A finds them admission-worthy only if B does not admit them, but B admits them no matter what A does. Each of the students in the intermediate range of scores, i.e., $[\underline{v}_A, \overline{v}_B]$, receives an admission from only one college. The students with scores $v \in [\underline{v}_B, \underline{v}_A]$ receive admissions only from B, since that college alone finds them admission-worthy given that they are not admitted by A. This pattern of strategic targeting — i.e., forgoing good students sought after by the other college but admitting less attractive ones neglected by others — stands in stark contrast with the cutoff strategy equilibrium found by the existing literature (see Chade, Lewis and Smith, 2011).

The particular pattern of strategic targeting, namely how the two colleges coordinate exactly on the students in $[\underline{v}_A, \overline{v}_B]$, is indeterminate, and the figure depicts one possible coordination.[9] In practice, it is implausible for colleges to achieve the kind of precise coordination described in the pure-strategy equilibria. It seems much more plausible for colleges to randomize its admission over students with the intermediate range of scores $v \in [\check{v}, \hat{v}]$, where $\check{v} := \max\{\underline{v}_A, \underline{v}_B\}$ and $\hat{v} := \min\{\overline{v}_A, \overline{v}_B\}$.[10] A typical mixed-strategy equilibrium is depicted in Figure 3.3.

Notice that the admission strategies outside the intermediate range is similar to that in the above pure-strategy equilibrium, as completely pinned down by Theorem 1. For the intermediate range of scores, an interior fraction of the students are chosen to keep each college indifferent, as follows. For each $v \in [\check{v}, \hat{v}]$, let $\alpha(v) = \alpha_0(v)$ and $\beta(v) = \beta_0(v)$, where $H_A(v, \beta_0(v)) = 0$ and $H_B(v, \alpha_0(v)) = 0$, or equivalently,

$$\alpha_0(v) := \frac{v - \lambda\, \hat{s}_B}{v\, \overline{\mu} - \lambda\, \hat{s}_B\, \mu_-(\hat{s}_B)} \tag{3.2}$$

and

$$\beta_0(v) := \frac{v - \lambda(1 - \hat{s}_A)}{v(1 - \overline{\mu}) - \lambda(1 - \hat{s}_A)(1 - \mu_+(\hat{s}_A))}. \tag{3.3}$$

One can easily check that $\alpha_0(v), \beta_0(v) \in [0, 1]$ for $v \in [\check{v}, \hat{v}]$. If college B admits a fraction $\beta_0(v)$ of students with v, then college A's marginal gain from admitting those students is zero, so it is indifferent about admitting them. Hence, it is college A's best response to randomize according

[9] As noted, there may be many ways for colleges to coordinate their admissions for students with $v \in [\check{v}, \hat{v}]$, where $\check{v} := \max\{\underline{v}_A, \underline{v}_B\}$ and $\hat{v} := \min\{\overline{v}_A, \overline{v}_B\}$. The range of different pure-strategy equilibria can be summarized by two extreme types of equilibria. We call a competitive equilibrium an **A-priority equilibrium** if $\alpha(v) = 1$ for all $v \in [\check{v}, \hat{v}]$, and a **$B$-priority equilibrium** if $\beta(v) = 1$ for all $v \in [\check{v}, \hat{v}]$. In words, in an i-priority equilibrium, the coordination is tilted in favor of college i. Clearly, between these two equilibria, one can construct (infinitely) many equilibria.

[10] It is important to note that the thresholds are not necessarily the same as in the pure-strategies, since different equilibria involve different cutoff states, $(\hat{s}_A, \hat{s}_B)$, which affect the marginal payoff functions H_A and H_B.

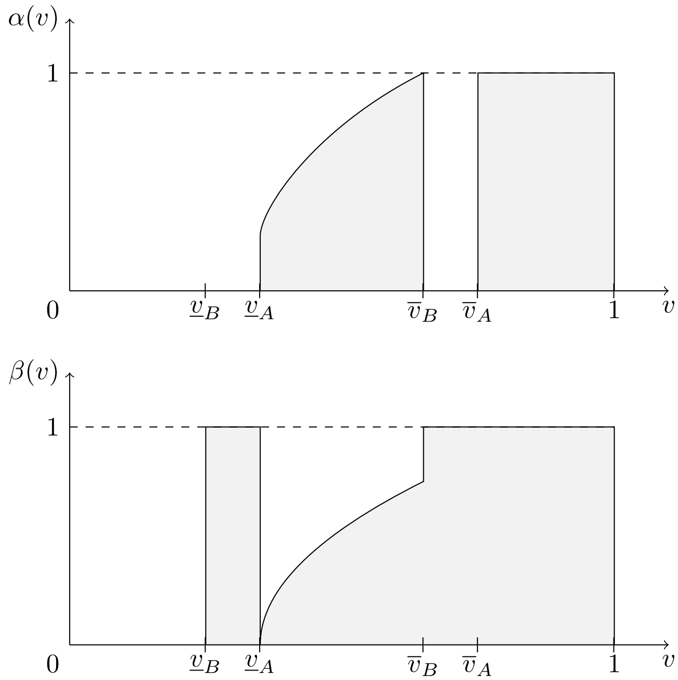


Figure 3.3: Mixed-Strategy Equilibrium

to $\alpha_0(\cdot)$. Since $H_B(v, \alpha_0(v)) = 0$, college B is indifferent, making its randomization a best response. Observe that both $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ are increasing in v, which means that colleges admit a higher fraction of students with higher scores. This is intuitive: Higher score students are more valuable all else equal, so admitting a higher fraction of those students is necessary to keep the opponent college indifferent. It is also interesting to observe discrete jumps in this figure — $\alpha_0(\underline{v}_A) > 0$ and $\beta_0(\overline{v}_B) < 1$. The former follows from the fact that $\underline{v}_A > \underline{v}_B$ which implies $H_B(\underline{v}_A, 0) > 0$, and the latter follows from $\overline{v}_A > \overline{v}_B$ which implies $H_A(\overline{v}_B, 1) < 0$.

There could be many ways for colleges to play mixed-strategies: For instance, colleges could coordinate to use a pure-strategy for some students, say $[\tilde{v}, \hat{v}]$ for some $\tilde{v} \in (\check{v}, \hat{v})$, and use mixed-strategies for $v \in [\check{v}, \tilde{v}]$. Consistent with our selection, we focus on the **maximally mixed equilibrium** (MME, in short) in which both colleges play mixed-strategies (α_0, β_0) for students with $v \in [\check{v}, \hat{v}]$ and according to Theorem 1 for outside that range.

The characterization of equilibria has so far rested on the necessary conditions for competitive equilibria, particularly the "local" incentive compatibility with respect to each type of students. Whether the preceding characterization based on MME admits a well-defined strategy profile and, if so, whether it constitutes competitive equilibria are not clear. We shall address these issues in the next subsection.

Before proceeding, though, it is important to recognize that a randomization by each college arises from its attempt to avoid competition for students in the intermediate range of scores. In this sense, as long as a competitive equilibrium admits the intermediate region, i.e., if $\check{v} < \hat{v}$, one can

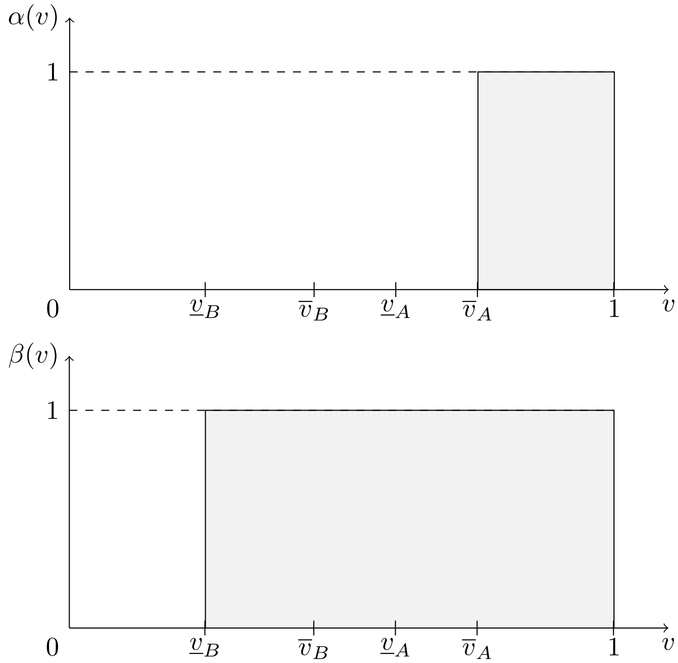


Figure 3.4: Cutoff Equilibrium

say that equilibrium involves strategic targeting, regardless of whether the colleges play a mixed- or a pure-strategy. We say *competitive equilibrium exhibits **strategic targeting*** if $\check{v} < \hat{v}$.

When do competitive equilibria exhibit strategic targeting? Note that Theorem 1 does not preclude a competitive equilibrium in which $\hat{v} < \check{v}$. Figure 3.4 depicts such a possibility with $\underline{v}_B < \overline{v}_B < \underline{v}_A < \overline{v}_A$. As before, college i admits students with $v > \overline{v}_i$ and rejects those with $v < \underline{v}_i$. Observe that college A does not admit any student with $v \in [\underline{v}_A, \overline{v}_A]$, since college B admits them for sure (because $\overline{v}_B < \underline{v}_A$). Even though colleges have targeting incentives in this example, the resulting equilibrium is indistinguishable from the cutoff equilibria featured in the existing research.

A natural question is when such an equilibrium can be ruled out. The exact condition for its existence appears difficult to find, but we show next that the symmetric environment is sufficient to guarantee strategic targeting behavior.

Theorem 2. *If the environment is symmetric (i.e., $\mu(s) = 1 - \mu(1-s)$ for all s), then every competitive equilibrium exhibits strategic targeting.*

Proof. See Appendix A.4. ■

3.1 Existence of MME

We now show that there exists an equilibrium in which $\alpha(\cdot)$ and $\beta(\cdot)$ involve maximal mixing.[11]

[11] In the same proof, we also establish the existence of A- or B-priority equilibrium. Note that a general equilibrium existence follows from the Glicksberg-Fan theorem, since each college's strategy space is compact and convex, and

Theorem 3. *There exists a competitive equilibrium with maximal mixing.*

Proof. See Appendix A.5. ■

We sketch the proof here. The proof constructs equilibrium strategies $\alpha(\cdot)$ and $\beta(\cdot)$ of the desired property (i.e., maximal mixing) in terms of threshold states $(\hat{s}_A, \hat{s}_B)$. Since the latter space is Euclidean (whereas the former is functional), we can simply appeal to the Brouwer's fixed point theorem to establish the existence. To begin, fix any candidate threshold states $(\hat{s}_A, \hat{s}_B)$ for the two colleges. Next, we can construct the colleges' mutual best-responses (α, β) corresponding to the chosen $(\hat{s}_A, \hat{s}_B)$, by simply consulting the signs of H_A and H_B:

$$\alpha(v;\hat{s}) = \begin{cases} 1 & \text{if } H_A(v,1;\hat{s}) > 0 \\ 0 & \text{if } H_A(v,1;\hat{s}) < 0, \ \ H_B(v,1;\hat{s}) > 0 \\ \alpha_0(v;\hat{s}) & \text{if } H_A(v,1;\hat{s}) < 0 < H_A(v,0;\hat{s}), \ \ H_B(v,1;\hat{s}) < 0 < H_B(v,0;\hat{s}) \\ 1 & \text{if } H_A(v,0;\hat{s}) > 0, \ \ H_B(v,0;\hat{s}) < 0 \\ 0 & \text{if } H_A(v,0;\hat{s}) < 0 \end{cases}$$

where $\alpha_0(\cdot)$ satisfies $H_B(v, \alpha_0(v)) = 0$ for $v \in [\check{v}, \hat{v}]$ as given by (3.2), and $\beta(v;\hat{s})$ is defined analogously.

Note that the construction pieces together the implications of Lemma 2, much as we did before, except that, for the intermediate v's, we require each college to mix in such a way to keep the opponent indifferent. In this way, a given $(\hat{s}_A, \hat{s}_B)$ pins down the maximally mixing mutual best responses (α, β). Since the threshold states $(\hat{s}_A, \hat{s}_B)$ are arbitrary, there is no guarantee that the constructed strategies reproduce them as the correct thresholds. In fact, they will reproduce another possible threshold states $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$:[12]

$$\tilde{s}_A = \inf\{s \in [0,1] \,|\, m_A(s;\hat{s}) - \kappa > 0\} \quad \text{and} \quad \tilde{s}_B = \inf\{s \in [0,1] \,|\, m_B(s;\hat{s}) - \kappa > 0\}, \tag{3.4}$$

where m_A and m_B are derived from the formulae (2.1) and (2.2).

But this process defines a mapping $T : [0,1]^2 \to [0,1]^2$ such that $T(\hat{s}) = \tilde{s}$. In Appendix A.5, we apply the Brouwer's fixed point theorem to show that T admits a fixed point $\hat{s}^* = (\hat{s}^*_A, \hat{s}^*_B)$ such that $T(\hat{s}^*) = \hat{s}^*$. Clearly, the strategies (α, β) constructed as above based on this fixed point $\hat{s}^* = (\hat{s}^*_A, \hat{s}^*_B)$ does form mutual best responses for the colleges, given the accurate thresholds.

Note that the strategies (α, β) thus found form best responses in the "local" sense: (α, β) entails

each college's payoff function is concave in its own strategy. That is, if one does not insist on the particular structure of behavior we impose on MME (or A- or B-priority), it is easy to show the existence of equilibrium admission strategies.

[12] As usual, these formulae are valid only if the associated sets in (3.4) are nonempty. If they are empty, then threshold values are set equal to one for $\tilde{s}_A$ and zero for $\tilde{s}_B$.

no incentive for each college to deviate in its admission decisions on type-v students, for each v, *holding constant its own admission strategies with respect to the other students.* A college may still profitably deviate on a mass of students. To show that no such global deviation is profitable, we consider a variation of $\alpha(\cdot)$ such that for any $t \in [0,1]$,

$$\alpha(v;t) := t\tilde{\alpha}(v) + (1-t)\alpha(v), \tag{3.5}$$

where $\tilde{\alpha}(v) \in [0,1]$ is an arbitrary strategy. We then define A's payoff function in terms of $\alpha(v;t)$, $V(t) := \pi_A(\alpha(v;t))$. Observe that $\pi_A(\tilde{\alpha}) = V(1)$ and $\pi_A(\alpha) = V(0)$. Therefore, the proof is completed by showing that $V(1) \leq V(0)$. Because $\tilde{\alpha}(\cdot)$ is arbitrary, this proves that $\alpha(\cdot)$ is a best response for a given $\beta(\cdot)$. See Appendix A.5 for details.

3.2 Properties of Equilibria

We have seen that the equilibrium outcome involves strategic targeting. We now consider the properties of the equilibria in welfare and fairness.

Let us first define assignment and outcome. For each state s, an **assignment** is a mapping from $\mathcal{V} \times \{A, B\}$ into the fraction of students assigned to each college. That is, an assignment is an allocation of the types of students in terms of scores and preferences to the colleges. An **outcome** is a mapping from a state to an assignment, i.e., the realized allocation in state s.

We say that a student has a **justified envy** at state s if at that state she prefers a college to the one she enrolls in, even though the former enrolls a student with a lower score. An outcome is said to be **fair** if for almost every state, the assignment it selects has no justified envy for almost all students. Next, an outcome is **Pareto efficient** if for almost every state, the assignment it selects is not Pareto dominated, i.e., there is no other assignment in which both colleges and all students are weakly better off and either at least one college or a positive measure of students is strictly better off relative to the initial assignment.

It is also useful to study the welfare of one side, taking the other side simply as resources. We say that an outcome is **student efficient** if for almost every state, there is no other assignment in which all students are weakly better off and a positive measure of students is strictly better off relative to the initial assignment that the outcome selects. An outcome is said to be **college efficient** if for almost every state, no other assignment can make both colleges weakly better off and at lease one college strictly better off relative to the assignment that the outcome selects. Notice that even if an outcome is Pareto efficient, it need not imply student efficiency or college efficiency.

The next theorem states properties of equilibria that arise in decentralized matching.

Theorem 4. *(i) Any non-competitive equilibrium is unfair, student inefficient, but college efficient.*

(ii) Any non-competitive equilibrium is Pareto inefficient unless almost every student admitted by one college has higher score than those admitted by the other college.

(iii) Any competitive equilibrium is student, college and Pareto inefficient.

(iv) Any competitive equilibrium is unfair if and only if it exhibits strategic targeting.

Proof. See Appendix A.6 ■

4 Multidimensional Performance Measures and Evaluation Distortion

In the baseline model, we have assumed that colleges assess students based on the common performance measure. In practice, colleges consider multiple dimensions of students' qualities and performances, academic as well as non-academic. Some performance dimensions are more common among colleges; for instance, the SAT scores or grade points average of students are commonly observed and interpreted virtually the same by colleges. Others are less correlated. For instance, many colleges require college-specific essays and testing (e.g., those in Korea and Japan). Non-academic performance measures are particularly rich and difficult to quantify, and colleges are likely to focus on different aspects and interpret them differently. For instance, some colleges may pay attention to students' community service or leadership activities, whereas others may pay more attention to extracurricular activities such as musical or athletic talents. So colleges' evaluation of students on these dimensions are likely to be less correlated. We show that strategic targeting takes a particular form in this environment: Colleges bias their admissions criteria by placing excessive weight on non-common performances.

To this end, we extend our model as follows. A student's type is described as a triple $(v, e, e') \in \mathcal{V} \times E \times E' \equiv [0, 1]^3$, where v is distributed according to $G(\cdot)$ as before, and e and e' are conditionally independent on v and are distributed according to $X(\cdot|v)$ and $Y(\cdot|v)$, respectively, which admit densities $x(\cdot|v)$ and $y(\cdot|v)$. We also assume that $X_v(e|v) < 0$ and $Y_v(e'|v) < 0$ for all $e, e' \in [0, 1]$. That is, a student with higher v has a higher probability to have higher e and e'. We also assume full support of G, X, Y. College A only values (v, e) and college B only cares about (v, e'). Specifically, we assume that college A derives payoff $U(v, e)$ from matriculating student with type (v, e, e'), where U is strictly increasing and differentiable in both arguments. Likewise, college B realizes payoff $V(v, e')$ from matriculating the same type of student, where V is strictly increasing and differentiable in (v, e').

One interpretation is that v is an academic performance measure observed commonly to both colleges, and e and e' correspond to different dimensions of extracurricular activities that the two colleges focus on. Alternatively, v is a student's test scores of the nationwide exam, and e and e' may represent a student's performance on college-specific tests or interviews.

College A's strategy is now described as a mapping $\alpha : \mathcal{V} \times E \to [0, 1]$ with interpretation that it admits a fraction α of students with type (v, e). Likewise, college B's strategy is described by

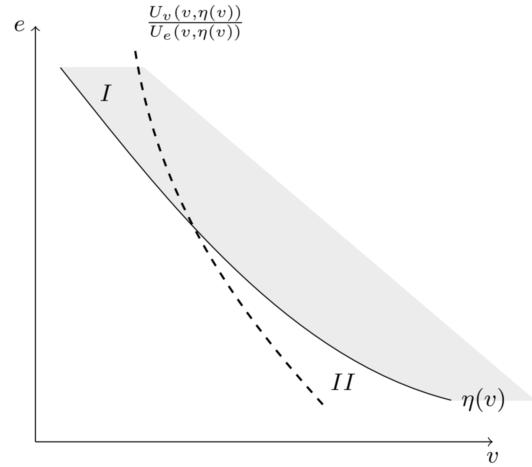


Figure 4.1: College's Cutoff Locus

a mapping $\beta : \mathcal{V} \times E' \to [0,1]$. The enrollment uncertainty facing A with regard to students with type (v, e) depends on whether those students receive an admission offer from B. But since e' is conditionally uncorrelated with e, the probability of such event is $\overline{\beta}(v) := \mathbb{E}_{e'}[\beta(v, e')|v]$. Likewise $\overline{\alpha}(v) := \mathbb{E}_e[\alpha(v, e)|v]$ is relevant for B to assess its enrollment uncertainty.

For given $\overline{\alpha}(\cdot)$ and $\overline{\beta}(\cdot)$, the mass of students enrolling in college A in state s is

$$m_A(s) = \int_0^1 \int_0^1 \alpha(v, e)[1 - \overline{\beta}(v) + \mu(s)\overline{\beta}(v)]\, dX(e|v)\, dG(v).$$

Hence, A's payoff is described as follow:

$$\begin{aligned} \pi_A &= \int_0^1 \int_0^1 U(v, e)\alpha(v, e)[1 - \overline{\beta}(v) + \overline{\mu}\overline{\beta}(v)]\, dX(e|v)\, dG(v) - \lambda\, \mathbb{E}_s[m_A(s) - \kappa \,|\, s > \hat{s}_A](1 - \hat{s}_A) \\ &= \int_0^1 \int_0^1 H_A(v, e, \overline{\beta}(v))\, dX(e|v)\, dG(v) + \lambda\, (1 - \hat{s}_A)\, \kappa, \end{aligned}$$

where

$$H_A(v, e, \overline{\beta}(v)) := U(v, e)[1 - \overline{\beta}(v) + \overline{\mu}\overline{\beta}(v)] - \lambda(1 - \hat{s}_A)\left(1 - \overline{\beta}(v) + \mu_+(\hat{s}_A)\overline{\beta}(v)\right). \tag{4.1}$$

We focus on a cutoff strategy equilibrium in which college A admits student type (v, e) if and only if $e \geq \eta(v)$ for some η nonincreasing in v, and college B admits student type (v, e') if and only if $e' \geq \xi(v)$ for some ξ nonincreasing in v. For instance, the shaded area in Figure 4.1 depicts the types of students A admits under a cutoff strategy. Appendix A.7 provides a condition under which cutoff equilibrium exists. Such an equilibrium is quite plausible since the use of non-common performance measure by the colleges lessens their head-on competition and the associated enrollment uncertainty.

The question we focus on here is whether the colleges may further reduce the head-on competition and the enrollment uncertainty by placing more weight on the non-common performance measures relative to their common preferences. Consider college A. (College B's incentive will be analogous.) Inspection of A's preference makes it clear that under the cutoff equilibrium, it must accept student type (v, e) if and only if $H_A(v, e, \overline{\beta}(v)) \geq 0$. In particular, the cutoff locus $e = \eta(v)$ must satisfy $H_A(v, \eta(v), \overline{\beta}(v)) = 0$ whenever $\eta(v) \in (0, 1)$. Its slope $-\eta'(v)$ shows the "relative worth" of the student's common performance v in A's evaluation, as measured in the units of the student's non-common performance that A is willing to give up to obtain a unit increase in her common performance. The higher this value is, the higher weight A places on the common performance. In particular, we shall say that the college *under-weights a student's common performance* v *and over-weights her non-common performance* e if for all v,

$$-\eta'(v) \leq \frac{U_v(v, \eta(v))}{U_e(v, \eta(v))}$$

and the inequality is strict for a positive measure of v. Suppose for instance $U(v, e) = (1-\rho)v + \rho e$, then the condition means that $-\eta'(v) \leq \frac{1-\rho}{\rho}$, so the college places a weight less than $1 - \rho$ to common performance v and the weight more than ρ to non-common performance e.

Theorem 5. *In a cutoff equilibrium, each college under-weights a student's common performance and over-weights her non-common performance.*

Proof. See Appendix A.7. ■

The distortion of the evaluation criterion makes the equilibrium outcome unfair since justified envy arises in a positive measure of states for those students with (v, e, e') who prefer A to B (those who are in the area II at the bottom between dotted and solid lines in Figure 4.1) and should have outranked some of those students admitted by A (those who are in the area I at the top between solid and dotted lines). In equilibrium, a college is also underfilled in a positive measure of states, and assigning unmatched students to those unfilled seats improves the social welfare without hurting any other students or colleges.

Theorem 6. *The cutoff equilibrium is unfair and students, college and Pareto inefficient.*

5 Coordinated Matching: Self-Targeting

So far, we have characterized the pattern of colleges' strategic targeting and provided existence and welfare and fairness properties of such equilibria. In the current and the following sections, we study two common ways for colleges to alleviate their yield management burden in decentralized matching. We consider the case that students' type is single dimensional as in the baseline model.

One common method used in many countries is to limit the number applications that students can submit. For instance, students cannot apply to both Cambridge and Oxford in the UK, and applicants in Japan can only apply to one public university.[13] In Korea, all schools (more precisely, college-department pairs) are partitioned into three groups and students are allowed to apply to only one in each group.[14]

Limiting the application forces students to "self-target" colleges. Since students are likely to target schools they are most likely to accept when admitted, this method improves the odds of enrollment for colleges and thus their yield management burden. In our model with two colleges, if the number of applications is restricted to one, colleges face no enrollment uncertainty because no student admitted by a college will turn down its offer; so their admission behavior is non-strategic; namely, they admit students in the order of v until their capacities are filled. However, students' application behavior will be strategic; thus, the overall welfare effects are not clear a priori.

We now provide a simple model showing students' application behavior when the students can apply to only one of the two colleges. To this end, we introduce students' cardinal preferences for colleges.[15] Each student has a taste $y \in \mathcal{Y} \equiv [0,1]$, which is independent of score $v \in [0,1]$. A student with taste y obtains payoff y from attending college A and $1-y$ from attending college B. Thus, students with $y \in [0, \frac{1}{2}]$ prefer B to A, and those with $y \in [\frac{1}{2}, 1]$ prefer A to B. To facilitate the analysis, we assume that colleges observe an applicant's score v but not her preference y, while each student knows her preference y but not her score v.[16] In reality, even though students submit their records to colleges, they do not know precisely how they are ranked by colleges. See Avery and Levin (2010) for the same treatment.

A student's taste y is drawn according to a distribution that depends on the underlying state. For a given s, let $K(y|s)$ be the distribution of y with a density function $k(y|s)$. Then, $\mu(s) \equiv 1 - K(\frac{1}{2}|s)$ is the mass of students who prefer A to B in state s. We assume that $\mu(s) \neq \frac{1}{2}$ for almost all s and that $k(y|s)$ is continuous and obeys (strict) monotone likelihood ratio property (MLRP), i.e., for any $y' > y$ and $s' > s$,

$$\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)}, \tag{5.1}$$

[13] Public colleges in Japan may hold three exams. The first one is called "zenki(former period)-exam" and the last one is called "koki(later-period)-exam". There are very small number of schools that have exam between these two exams. Students can apply to at most one public school at each exam date but the deadline for registering to the school that a student is admitted at zenki-exam is earlier than the date for applying the koki-exam.

[14] Although there is no such restriction in the US, high application fees may serve this role. See Chade and Smith (2006) and Chade, Lewis and Smith (2011) for students application decisions subject to application costs, without aggregate uncertainty.

[15] Note that this does not alter the previous analyses, because even if students have cardinal preferences, it is still a weak dominant strategy for students to apply to both colleges in the previous model.

[16] This also does not alter the previous analyses, because if students do not know their scores perfectly, then it is a strict dominant strategy for them to apply to both colleges when there is no restriction on the number of applications (see footnote 7).

meaning that a student's taste is more likely to be high in a high state. We further assume that there is $\delta > 0$ such that $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$ for any $y \in [0,1]$ and $s \in [0,1]$, which means that students' tastes change moderately according to states. Each student with taste y forms a posterior belief about the states,

$$l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s)ds}.$$

Before proceeding, we make the following observations: First, for the students, applying to a school dominates not applying at all. Second, since students do not know their scores and their preferences are independent of the scores, students' applications depend only on their preferences. Third, since students' preferences depend on states, the mass of students applying to each college varies across states. Let $n_i(s)$ be the mass of students who apply to college $i = A, B$ in state s.

Consider colleges' admissions decisions. Since a college faces no enrollment uncertainty, a cutoff strategy is optimal. If $n_i(s) \geq \kappa$ in state s, then college i will set its cutoff so as to admit students up to its capacity. Otherwise, it will admit all applicants. More precisely, the cutoff of college i in state s, denoted by $c_i(s)$, is given by

$$c_i(s) := \inf \{c \in [0,1] \,|\, n_i(s)[1 - G(c)] \leq \kappa\}.$$

Consider now students' application decisions. Fix any $\sigma : \mathcal{Y} \to [0,1]$ which maps from taste to a probability of applying to A. This induces the mass of students applying to A in each state s,

$$n_A(s) := \int_0^1 \sigma(y)k(y|s)\, dy.$$

Clearly, $n_B(s) = 1 - n_A(s)$. A student with taste y has a probability of being admitted by i

$$P_i(y|\sigma) = \mathbb{E}_s[1 - G(c_i(s)) \,|\, y, \sigma] = \int_0^1 q_i(s|\sigma)l(s|y)ds,$$

where $q_i(s|\sigma) := \min\left\{\kappa/n_i(s|\sigma), 1\right\}$ for $i = A, B$. Note that a student with taste y will apply to A if and only if

$$yP_A(y|\sigma) \geq (1-y)P_B(y|\sigma).$$

or equivalently,

$$T(y|\sigma) := yP_A(y|\sigma) - (1-y)P_B(y|\sigma) \geq 0.$$

Lemma 3. *Suppose $\delta \leq \frac{1}{2}$. In any equilibrium, there exists a cutoff $\hat{y}$ such that students with $y \geq \hat{y}$ apply to A and those with $y < \hat{y}$ apply to B. And such an equilibrium exists.*

Proof. See Appendix A.8. ■

Let $\hat{y}$ be the cutoff in the equilibrium. Since all students with $y \geq \hat{y}$ apply to A, the mass of

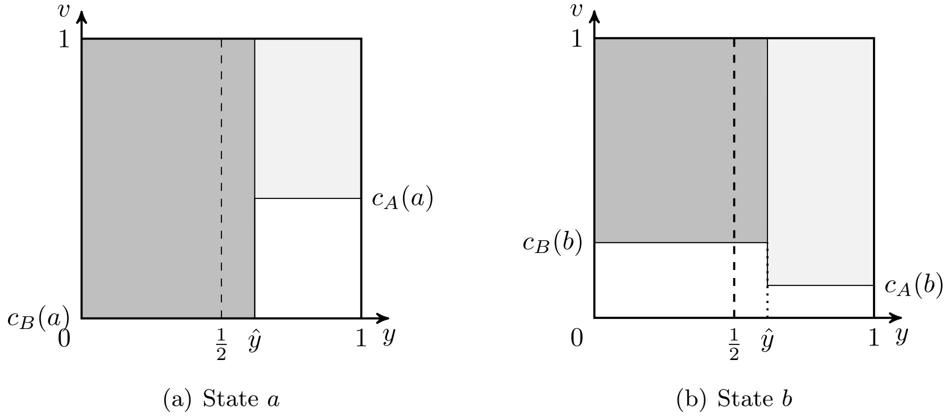


Figure 5.1: Equilibrium Assignment when $\kappa = 0.4$

students applying to A is $n_A(s) = \int_{\hat{y}}^{1} k(y|s)dy = 1 - K(\hat{y}|s)$, and similarly $n_B(s) = K(\hat{y}|s)$.

Theorem 7. *Suppose $\mu(s) > \frac{1}{2}$ $\left(\mu(s) = \frac{1}{2}\right)$ for almost all s. Then, $\hat{y} \in (\frac{1}{2}, 1)$ $\left(\hat{y} = \frac{1}{2}\right)$, where $\hat{y}$ is the equilibrium cutoff.*

Proof. See Appendix A.8. ■

Theorem 7 shows students' strategic applications when college A is more popular than the other for all states. Consider a student with taste y who expects that $P_B(y) > P_A(y)$ since A is more popular than B. If she prefers B $(y < \frac{1}{2})$, then it is optimal for her to apply to B, obviously. If the student prefers A $(y \geq \frac{1}{2})$, then there is a trade-off since her payoff is higher if she can attend A over B, but she believes that she has a higher chance of admission to B. Thus, if she mildly prefers A, then she may apply to B instead of A. We provide a simple example with two states to illustrate the results. Figure 5.1 depicts the equilibrium assignments of the example.

Example 1. Suppose there are two states a and b with equal probability. Let $K(y|a) = y^2$, $K(y|b) = y$ and $\kappa = 0.4$. Then, we have

$\hat{y}$	$n_A(a)$	$n_B(a)$	$c_A(a)$	$c_B(a)$	$n_A(b)$	$n_B(b)$	$c_A(b)$	$c_B(b)$
0.547	0.701	0.299	0.429	0	0.453	0.547	0.116	0.269

Observe that if $n_i(s) \geq \kappa$ for all s and all $i = A, B$, then the self-targeting eliminates colleges' yield management problem, since each college fills its capacity with the best students among those who applied to it. However, it does not hold in general because there can be under-subscription to a college in some state. In the above example, for instance, the mass of applicants to college B in state a is smaller than its capacity ($n_B(a) = 0.299 < \kappa = 0.4$).

Let us now consider welfare and fairness properties of the equilibrium outcome. First, the equilibrium is unfair. That is, justified envy arises in that (i) students who happen to have applied to a more popular school for a given state may be unassigned even though their scores would have been good enough for the other school (the area on the bottom right below the right shaded area of Figure 5.1(a)); and (ii) students who mildly prefer the popular school to the less popular one may be assigned to the latter college even though they could have been assigned to the popular one (the dark shaded area between $\frac{1}{2}$ and $\hat{y}$ of Figure 5.1(b)).

Second, there can be under-subscription to a college in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of a college, both the students and college will be better off. Thus, the equilibrium outcome is still student, college and Pareto inefficient.

In the next theorem, we provide conditions under which justified envy among students and/or under-subscription to a college arise.

Theorem 8. *The outcome is unfair. Suppose $K(\hat{y}|s) < \kappa$ for a positive measure of states. Then, college B suffers from under-subscription, and the outcome is student, college and Pareto inefficient.*

Proof. See Appendix A.8. ■

6 Wait-listing

In this section, we consider wait-listing as another way to cope with enrollment uncertainty. According to this method, a college would admit initially some applicants and wait-list some others and later admit students from the latter group when some of the former group decline admissions, and this process may repeat. Wait-listing is adopted by most colleges in France, Korea, and the US.[17] Typically, the acceptance decisions are not deferred or the number of iterations is limited. Hence, even though wait-listing entails more admission offers and acceptances than the baseline model or self-targeting, it does not fully eliminate congestion. For this reason, strategic targeting remains an issue as well.

To see this, we consider a simple extension of our baseline model. There are three colleges, A, B and C, each with a mass $\kappa < \frac{1}{3}$ capacity. There is a unit mass of students with score v, where v is distributed over $[0, 1]$ according to $G(\cdot)$ as before. All students like A and B better than C, but C is sufficiently better than not attending any school. Colleges' preferences are given by students' scores, but for each student, there is a probability ε that each of colleges A and B finds that the student is unacceptable. College C simply likes students according to their scores.

There are two states, a and b, with equal probability. In state i, $i = a, b$, a fraction s_i of students gets utility u from A and u' from B, and the remaining $1 - s_i$ students have the opposite preference, where $s_a = 1 - s_b > \frac{1}{2}$. In either state, students get utility u'' from C, where $u > u' > u''$ and

[17] In the US, nearly 45 percent of four-year colleges utilize wait lists in 2011 (Clinedinst, Hurley and Hawkins, 2012).

$u'' > (1-\varepsilon)u$. The latter assumption means that the certain utility from college C of a student is greater than the uncertain utility from the better school. Note that in state a, the mass of students who prefer A to B is larger than that of those who prefer B to A ($s_a > \frac{1}{2} > 1 - s_a$), and in state b, the former is smaller than the latter ($s_b < \frac{1}{2} < 1 - s_b$).

Suppose the capacity cost is prohibitively high so that at each time when a college makes admission decisions, it must be sure that the capacity will never be violated. The wait-listing has the following feature. In each round, the colleges make admission offers to a set of students and wait-list the remaining. The students who received offer(s) from college(s) must decide to accept or reject the offer immediately; that is, the acceptance decision cannot be deferred. After the first round, colleges A and B learn the state, so the game effectively ends in two rounds.

We show that there is no symmetric equilibrium in which both colleges A and B use a cutoff strategy (i.e., admit the top κ students among those who are acceptable) in the first round.

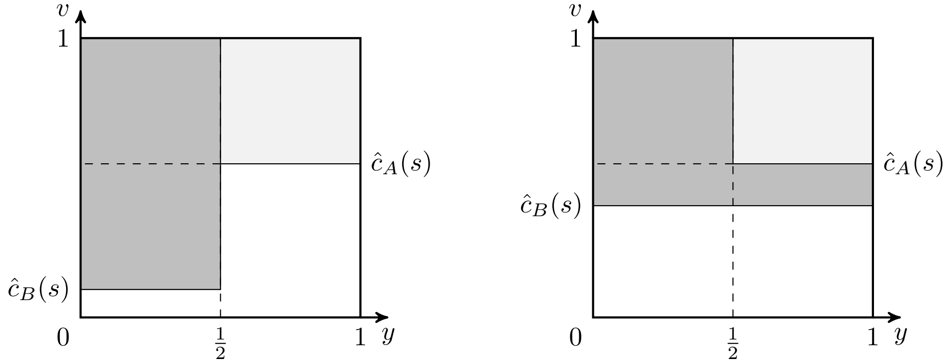
Theorem 9. *There is no symmetric equilibrium in which both A and B offer admissions to the top κ students (excluding those whom they find unacceptable) in the first round.*

Proof. See Appendix A.9. ■

Suppose colleges A and B consider making admission offers to the most preferred candidates up to their capacities with a plan to approach the next best students in case some of those first group students turn their offers down. The problem, however, is that when they are turned down by some of the first group students, they may not have the second best students available. The reason is that those latter group students are uncertain about whether A or B find them acceptable, hence if they receive an admission offer from college C, they may accept it. This in turn creates uncertainty for colleges on the students that remain after the first round. In particular, the students who remain after the first round are likely to be far worse than the second best group. This means in that in any symmetric equilibrium, each of A and B must directly offer admissions to some of those second group (at least with a positive probability) instead of some of the first group students. In other words, strategic targeting must occur in any symmetric equilibrium. The strategic targeting here can be traced to the uncertainty facing the colleges about what students will remain after each round. This uncertainty in turn arises from the uncertainty facing the students about whether better offers will emerge in the next round by turning down the current offer. Without the deferral of decisions, either by colleges in admitting students or by students in accepting offers, the uncertainty results in strategic targeting.[18]

Even though our characterization is partial, we have shown that strategic targeting must be part of the symmetric equilibrium. This means that the equilibrium outcome must be unfair. The equilibrium must be also student inefficient because there are two groups of students, among those

[18] As will be seen in the next section, the deferral of decisions allowed in the Gale-Shapley's algorithm solves this problem.



(a) Round 1

(b) Round 2

Figure 7.1: Deferred Acceptance Algorithm

in the second best group, one preferring A but assigned to B and the other preferring B but assigned to A. Again, strategic targeting has undesirable consequences.

7 Centralized Matching via Deferred Acceptance

The most systemic way to respond to the enrollment uncertainty would be to centralize the matching procedure. College admissions are also centralized in some countries, such as Australia, China, Germany, Taiwan, Turkey and the UK.[19] In this section, we consider a centralized matching with a Gale and Shapley's Deferred Acceptance algorithm (henceforth DA). Not only is the DA employed in many centralized markets, such as public school admissions and medical residency assignments, but it has a number of desirable properties compared with the outcomes of decentralized matching, as we shall highlight below.

Suppose that the matching is organized by a clearinghouse that applies Gale and Shapley's student-proposing DA.[20] The algorithm works as follows. Initially, students and colleges report their preference orderings to the clearinghouse. In each round, students propose to the best schools that have not yet rejected them. The colleges then accept tentatively the applicants in the order of their scores up to their capacities and rejects the rest. This process is repeated until no further proposals are made, in which case each student is assigned to a college that has tentatively accepted her proposal.[21]

Figure 7.1 illustrates the process for the case $\mu(s) \geq \frac{1}{2}$. In the first round, a fraction $\mu(s)$ of students proposes to college A, and the remaining students propose to college B. Each college

[19]See Chen and Kesten (2011) for Shanghai mechanism and Westkamp (2012) for Germany medical school matchings.

[20]The outcome of college-proposing DA is the same as that of student-proposing DA in our model, since colleges have a uniform rank on students.

[21]Abdulkadiroğlu, Che and Yasuda (2012) and Azevedo and Leshno (2012) provide a model of DA in which a continuum mass of students is matched to a finite number of schools.

tentatively admits the top κ students among the applicants. Thus, colleges' cutoffs in this round, denoted by $\hat{c}_i(s)$, $i = A, B$, satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $(1 - \mu(s))[1 - G(\hat{c}_B(s))] = \kappa$ (see Figure 7.1(a)). Unassigned students then propose to another college at the second round, and again, colleges reselect the top κ students from those tentatively admitted and from the new applicants. Thus, colleges' cutoffs in this round satisfy $\mu(s)[1 - G(\hat{c}_A(s))] = \kappa$ and $1 - G(\hat{c}_B(s)) = 2\kappa$ (see Figure 7.1(b)). Since there are no more colleges to which unassigned students can apply, the assignment is finalized in the second round in our model.

Consider now the equilibrium properties of the DA outcome. Under DA, the matching is strategy proof for the students, so the students have a dominant strategy of reporting their preferences truthfully (Dubins and Freedman, 1981; Roth, 1982). In addition, colleges in our model also report their rankings and capacities truthfully in a Nash equilibrium.

Lemma 4. *Given the common college preferences, it is an ex post equilibrium for colleges to report their rankings and capacities truthfully.*

Proof. See Appendix A.10. ■

The matching in the equilibrium involves no justified envy (Gale and Shapley, 1962; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) and is efficient among students (because colleges' preferences are acyclic in the sense of Ergin (2002)) and Pareto efficient (an implication of stability). It also eliminates colleges' yield management problem completely. Colleges never exceed their capacities (because it is never allowed by the algorithm) and have no seats left unfilled in the presence of acceptable unmatched students (a consequence of stability).

In fact, given the homogeneous preferences of the colleges, there exists a single cutoff such that a student is assigned to a college under DA if and only if her score exceeds that cutoff. In order words, only those with the top 2κ scores are assigned. This outcome is jointly optimal for the two colleges, in the sense that if the two colleges were to merge, it will choose exactly top 2κ students in each state. In contrast, a competitive equilibrium in decentralized matching entails unfilled seats for colleges in low-demand states and exceeded quotas in high-demand states, so the assignment is far from jointly optimal. This observation suggests that at least one college must be strictly better off from a shift from decentralized matching to centralized matching via the deferred acceptance algorithm. Despite the overall benefit from switching centralization via DA, it is possible for one college to be worse off. To see this, consider the following example.

Example 2. Let $v \sim U[0, 1]$, $\lambda = 5$, $\kappa = 0.45$ and $\mu(s) = \frac{2}{5}s + \frac{3}{5}$. Then, in a decentralized admission, there is a MME such that $\check{v} = \underline{v}_A < \overline{v}_B = \hat{v}$ and colleges' payoffs in the equilibrium are $\pi_A = 0.283$ and $\pi_B = 0.180$. Suppose now that the DA is in use. Then, their payoffs are $\pi_A^{DA} = 0.321$ and $\pi_B^{DA} = 0.174$. Notice that $\pi_A^{DA} + \pi_B^{DA} = 0.495 > \pi_A + \pi_B = 0.463$ (overall benefit for the two colleges), $\pi_A^{DA} > \pi_A$ (college A is strictly better off), but $\pi_B^{DA} < \pi_B$ (college B is worse off).

In this example, college A is more popular than B for all states. Yet, in a decentralized matching, strategic targeting enables college B to attract top students who it would otherwise not be able to attract under DA. This may explain why centralized college admissions are not very common, for instance, in contrast with public high school admissions. In the latter, the schools are largely under the control of the school system which serves the interest of the students. In contrast, colleges are independent strategic players with their own interest to pursue.

Equilibrium properties of the outcome under DA are summarized in the follow.

Theorem 10. *Under DA, the equilibrium outcome is fair, Pareto and student efficient, and jointly optimal among the colleges. However, some college may be worse off relative to decentralized matching.*

8 Conclusion

The current paper has introduced and analyzed a new model of decentralized college admissions. In the model, colleges make admission decisions subject to aggregate uncertainty about students' preferences and linear costs for any enrollment exceeding the capacity. We find that colleges' admission decisions become a tool for strategic yield management and in equilibrium, colleges try to reduce their enrollment uncertainty by strategically targeting students with their admissions. We also show that when colleges consider students' non-academic performance or extracurricular activities, the use of these aspects may lessen head-on competition among colleges. However, strategic targeting still entails as colleges placing over-weights on those non-common performance measure.

We also obtain the welfare and fairness implications of the equilibrium outcomes. We show that the equilibrium outcome under decentralized matching entails justified envy and is Pareto inefficient. Our analytical model also permits a clear comparison of the outcomes that would arise when students are forced to self-target (by the limited set of schools they can apply to), when admissions are made sequentially, and when the market is centralized via DA. Both self-targeting and wait-listing alleviate colleges' yield management burden, but strategic targeting and enrollment uncertainty remain. Thereby, so do inefficiencies and justified envy. Centralized matching via DA completely eliminates the yield management problem and justified envy, and it also achieves Pareto efficiency. At the same time, not all colleges may benefit from such a centralized matching. This last observation may explain why college admissions remain decentralized in many countries.

Our analyses yield implications on several issues:

Early Admissions. Early admissions are widely used in the US and Korea. In these countries, students can apply early often to a restricted set of schools, and the schools early-admit them (with binding or non-binding requirements for students to accept them). The remaining students and seats are then allocated through regular admissions, operating much as in our baseline model. The resulting process involves the sequencing of admissions, as studied in Section 6, and the restricted choice in the early round resembles self-targeting, as studied in Section 5. While the process is too complicated to model in our framework, particularly with aggregate uncertainty, our analyses suggest an important purpose the early admissions program serves. By restricting the number of applications, the early admissions programs induce students to reveal their preferences for colleges. This, together with the sequencing, allows colleges to forecast and manage the enrollment uncertainty more effectively than they could without the program. We believe this is an important function of the early admissions, in addition to those recognized by other recent literatures (Avery and Levin, 2010; Lee, 2009). Regardless of the motives, the programs restrict choices for students and force colleges to make decisions based on less than full information that will become available to them. As seen in Section 5 and in Avery and Levin (2010), students are likely to respond strategically, which will likely entail justified envy and inefficiencies.

Colleges' Preferences for Loyalty and Enthusiasm. It is well documented that colleges favor students who are eager to attend them. Students who convey seriousness of their interests through campus visits, essays, and webcam interviews are known to be marginally favored, especially by small liberal arts colleges. Early admissions, as Avery and Levin (2010) argue, also serve as a tool for colleges to identify enthusiastic applicants and favor them in the admission. Likewise, the "legacy" admits (who have a family history with the school) can be seen as a way for colleges to identify and favor those who have strong preferences. It is entirely plausible that these preferences by colleges are intrinsic, as postulated by Avery and Levin (2010). But, our theory suggests that such a preference by colleges could also arise endogenously from their desire to manage enrollment uncertainty. The main logic of our theory is that those who are more likely to accept a college's admission contributes less to enrollment uncertainty than those who are not as serious, suggesting that even a college with no intrinsic preference for the former students has a reason to favor them. In this sense, campus visits, essays and legacy admits all serve as a device for colleges to target those who are more likely to come.

Specialized Requirements and College Specific Investments. Colleges often have special requirements for their applicants to fulfill. These requirements range from specialized essay questions, college-specific entrance exams, to specialized admissions tracks requiring specific qualifications. For example, colleges in Korea admit a number of students through specialized tracks that require specific qualifications, such as foreign language skills, awards in contests in science, music, invention, or information technologies. Such requirements help colleges to identify students with serious interests. More demanding requirements encourage students to make college-specific investments. Our theory suggests that these investments serve as a means by which colleges can target and secure enrollment of students even in early stages.

References

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003. "School Choice: A Mechanism Design Approach." *American Economic Review*, 93(3): 729–747. 26

Abdulkadiroğlu, Atila, Yeon-Koo Che, and Yosuke Yasuda. 2012. "Expending 'Choice' in School Choice." Mimeo. 25

Avery, Chistopher, and Jonathan Levin. 2010. "Early Admissions at Selective College." *American Economic Review*, 100(5): 2125–2156. 6, 20, 28

Avery, Christopher, Andrew Fairbanks, and Richard Zeckhauser. 2003. *The Early Admissions Game: Joining the Elite.* Havard University Press. 2

Azevedo, Eduardo M., and Jacob D. Leshno. 2012. "A Supply and Demand Framework For Two-Sided Matching Market." Mimeo. 25

Balinski, Michel, and Tayfun Sönmez. 1999. "A Tale of Two Mechanisms: Student Placement." *Journal of Economic Theory*, 84(1): 73–94. 26

Burdett, Kenneth, Shouyong Shi, and Randall Wright. 2001. "Pricing and Matching with Frictions." *Journal of Political Econoy*, 109(5): 1060–1085. 6

Chade, Hector, and Lones Smith. 2006. "Simultaneous Search." *Econometrica*, 74(5): 1293–1307. 2, 6, 20

Chade, Hector, Gregory Lewis, and Lones Smith. 2011. "Student Portfolios and the College Admissions Problem." Mimeo. 2, 6, 12, 20

Chen, Yan, and Onur Kesten. 2011. "From Boston to Shanghai to Deferred Acceptance: Theory and Experiments on A Famila of School Choice Mechanisms." Mimeo. 25

Clinedinst, Melissa E., Sarah F. Hurley, and David A. Hawkins. 2012. *2012 State of College Admissions.* National Association for College Admissions Counseling. 2, 4, 23

Dubins, L. E., and D. A. Freedman. 1981. "Machiavelli and the Gale-Shapley Algorithm." *American Mathematical Monthly*, 88(7): 485–494. 26

Ergin, Haluk. 2002. "Efficient Resoure Allocation on the Basis of Priorities." *Econometrica*, 79(6): 2489–2497. 26

Gale, D., and L. Shapley. 1962. "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69(1): 9–15. 26

Lee, Sam-Ho. 2009. "Jumping the Curse: Early Contracting with Private Information in University Admissions." *International Economic Review*, 50(1): 1–38. 6, 28

Lizzeri, Alessandro, and Nicola Persico. 2001. "The Provision of Public Goods Under Alternative Electoral Incentives." *American Economic Review*, 91(1): 225–239. 6

Lizzeri, Alessandro, and Nicola Persico. 2005. "A Drawback of Electoral Competition." *Journal of the European Economic Association*, 3(6): 1318–1348. 6

Montgomery, James D. 1991. "Equilibrium Wage Dispersion and Interindustry Wage Differentials." *Quarterly Journal of Economics*, 106(1): 163–179. 6

Neiderle, Muriel, and Leeat Yariv. 2009. "Decentralized Matching with Aligned Preferences." mimeo. 6

Ok, Efe A. 2007. *Real Analysis with Economic Applications.* Princeton University Press. 59

Roth, Alvin E. 1982. "The Economics of Matching: Stability and Incentives." *Mathematics of Operations Research*, 7(4): 617–628. 26

Roth, Alvin E., and Xiaolin Xing. 1997. "Turnaround Time and Bottlenecks in Market Clearing: Decentralized Matching in the Market for Clinical Psycologists." *Journal of Political Economy*, 105(2): 284–329. 1, 5, 6

Westkamp, Alexander. 2012. "An Analysis of the German University Admissions System." *Economic Theory*, forthcoming. 25

A Appendix A: Proofs

A.1 Proof of Lemma 1

Claim 1. *Suppose $\mathcal{V}_{AB}$ has zero measure. Then, the following results hold.*

(i) $m_A(s) = m_B(s) = \kappa$ for all $s \in [0, 1]$.

(ii) Almost every student with $v \geq G^{-1}(1 - 2\kappa)$ receives a admission.

Proof. (i) Since $\mathcal{V}_{AB}$ is a measure zero set, $m_i(s)$, $i = A, B$, is constant across states. Suppose $m_i(s) < \kappa$. Then, college i can benefit by admitting some students with measure less than $\kappa - m_i(s)$. Similarly, if $m_i(s) > \kappa$, then it can benefit by rejecting some students with measure less than $m_i(s) - \kappa$.

(ii) Observe that $\mathcal{V}_A \cup \mathcal{V}_B$ cannot have a gap, otherwise a college can benefit by replacing a positive measure of low score students with the same measure of students in the gap. So, it must be a connected interval with $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$. Since $m_A(s) = m_B(s) = \kappa$ for all s by Part (i), this means that almost every top 2κ students are admitted. ■

Note that the proofs for Parts (i), (ii) and (iv) of the lemma for noncompetitive equilibrium follow from Claim 1. We thus consider competitive equilibrium in what follows. We prove in the sequence of Parts (i), (iii) and (ii).

Proof of Part (i). Consider a competitive equilibrium. Suppose $m_A(1) < \kappa$. Let college A admit a mass $\kappa - m_A(1)$ of students. Then, the mass of students attending A in this case, denoted by $\widetilde{m}_A(s)$, satisfies that for any $s < 1$,

$$m_A(s) < m_A(s) + \mu(s)[\kappa - m_A(1)] \leq \widetilde{m}_A(s) \leq m_A(s) + [k - m_A(1)] < \kappa,$$

where the first and the last inequality follow from the fact that $m_A(s) < m_A(1)$ for $s < 1$ (since $\mu(\cdot)$ is strictly increasing in s). Observe that A benefits from such deviation since it admits more students without having over-enrollment. Hence, we must have $\kappa \leq m_A(1)$ in equilibrium. Similarly, if $m_A(0) > \kappa$, then A can benefit by rejecting a mass $m_A(0) - \kappa$ of students. Therefore, we must have $m_A(0) \leq \kappa \leq m_A(1)$ in any competitive equilibrium. The proof for college B is analogous. ■

Proof of Part (iii). We consider college A here. The proof for college B will be analogous. Since $\mu(\cdot)$ is strictly increasing and continuous in s, so is $m_A(\cdot)$. Thus, there exists $\hat{s}_A \in [0, 1]$ such that $m_A(\hat{s}_A) = \kappa$ by Part (i). We show that $\hat{s}_A \neq 0, 1$ in what follows.

Suppose $\hat{s}_A = 0$. Then, $m_A(s) > m_A(0) = \kappa$ for all $s > 0$. Thus, college A's payoff is

$$\pi_A = \int_0^1 v\alpha(v)(1 - \beta(v) + \overline{\mu}\beta(v))dG(v) - \lambda \int_0^1 [m_A(s) - \kappa]ds,$$

where

$$m_A(s) = \int_0^1 \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v).$$

Let A reject a positive measure of students, say $(c, c+\delta) \in \mathcal{V}_A$. Then, its payoff is

$$\widetilde{\pi}_A = \int_{[0,1]\setminus(c,c+\delta)} v\alpha(v)(1-b(v)+\overline{\mu})\beta(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 [\widetilde{m}_A(s)-\kappa]ds,$$

where

$$\widetilde{m}_A(s) = m_A(s) - \int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \tag{A.1.1}$$

and $\tilde{s}_A$ is such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A = 0$ since $\widetilde{m}_A(s) < m_A(s)$. Now, we can choose δ such that $\tilde{s}_A < \varepsilon$ for sufficiently small $\varepsilon > 0$. Then, A's net payoff from the deviation is

$$\begin{aligned}
&-\int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\overline{\mu}\beta(v))dG(v) - \lambda\int_{\tilde{s}_A}^1 [\widetilde{m}_A(s)-\kappa]ds + \lambda\int_0^1 [m_A(s)-\kappa]ds \\
=&-\int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\overline{\mu}\beta(v))dG(v) + \lambda\int_{\tilde{s}_A}^1 [m_A(s)-\widetilde{m}_A(s)]ds + \lambda\int_0^{\tilde{s}_A} [m_A(s)-\kappa]ds \\
=&-\int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds - \int_0^{\tilde{s}_A}\Big(\int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds \\
&+\lambda\int_{\tilde{s}_A}^1\Big(\int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds + \lambda\int_0^1 [m_A(s)-\kappa]ds \\
=&\int_{\tilde{s}_A}^1\Big(\int_c^{c+\delta} (\lambda-v)\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds \\
&-\int_0^{\tilde{s}_A}\Big(\int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds + \lambda\int_0^{\tilde{s}_A} [m_A(s)-\kappa]ds \\
>&0,
\end{aligned}$$

where the second equality follows from (A.1.1) and the last inequality holds for sufficiently small ε.

Next, suppose $\hat{s}_A = 1$. Then, $m_A(s) < m_A(1) = \kappa$ for all $s < 1$. Let A admit all students in $(c, c+\delta) \notin \mathcal{V}_A$ for some $c < 1$. Then, the mass of students attending A becomes

$$\widetilde{m}_A(s) = m_A(s) + \int_c^{c+\delta} (1-\beta(v)+\mu(s)\beta(v))dG(v). \tag{A.1.2}$$

Let $\tilde{s}_A$ be such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A < \hat{s}_A = 1$ since $\widetilde{m}_A(s) > m_A(s)$. We can choose δ such that $1 - \tilde{s}_A < \varepsilon$ for sufficiently small ε. Then, A's net payoff from the deviation is

$$\begin{aligned}
&\int_c^{c+\delta} v(1-\beta(v)+\overline{\mu}\beta(v))dG(v) - \lambda\int_{\tilde{s}_A}^1 (\widetilde{m}_A(s)-\kappa)ds \\
=&\int_c^{c+\delta} v(1-\beta(v)+\overline{\mu}\beta(v))dG(v) - \lambda\int_{\tilde{s}_A}^1 \Big(m_A(s) + \int_c^{c+\delta} (1-\beta(v)+\mu(s)\beta(v))dG(v) - \kappa\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_c^{c+\delta} v(1-\beta(v)+\overline{\mu}\beta(v))dG(v) - \lambda \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} (1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds + \lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)] ds \\
&= \int_0^{\tilde{s}_A} \Big(\int_c^{c+\delta} v(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds - \int_{\tilde{s}_A}^1 \Big(\int_c^{c+\delta} (\lambda - v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds \\
&\quad + \lambda \int_{\tilde{s}_A}^1 [\kappa - m_A(s)] ds \\
&> 0
\end{aligned}
$$

where the first equality follows from (A.1.2) and the last inequality holds for sufficiently small ε. ■

Proof of Part (ii). We first show $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$ and then show that $\mathcal{V}_A \cup \mathcal{V}_B$ is a connected interval and $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$.

Step 1. $\sup\{\mathcal{V}_A \cup \mathcal{V}_B\} = 1$.

Proof . Suppose to the contrary that $\overline{c} := \sup\{\mathcal{V}_A \cup \mathcal{V}_B\} < 1$. We show that at least one college can benefit by rejecting some students in favor of those with $[\overline{c}, 1]$.

Suppose $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ contains an open interval with positive measure for some $i = A, B$. Then, it is clear that college i can benefit by rejecting a positive measure of students from the bottom of $\mathcal{V}_i \setminus \mathcal{V}_{AB}$ and admits the same measure of students from 1.

Suppose now it is not the case. Let a college, say A, reject students in $(c, c+\delta) \in \mathcal{V}_{AB}$ and admit those in $(1-\varepsilon, 1]$ instead, where δ and ε satisfy

$$
\int_{1-\varepsilon}^1 v dG(v) = \int_c^{c+\delta} v dG(v) \tag{A.1.3}
$$

and

$$
\int_{1-\varepsilon}^1 1 dG(v) = \int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(\hat{s}_A)\beta(v))dG(v), \tag{A.1.4}
$$

for given $\hat{s}_A$ such that $m_A(\hat{s}_A) = \kappa$. The mass of students attending A from this deviation is

$$
\widetilde{m}_A(s) = m_A(s) + \int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v).
$$

Note that $\widetilde{m}_A(\hat{s}_A) = m_A(\hat{s}_A)$. Denote A's payoff from the deviation by $\widetilde{\pi}_A$. Then, A's net payoff from the deviation, $\widetilde{\pi}_A - \pi_A$, is

$$
\begin{aligned}
&\int_{1-\varepsilon}^1 v dG(v) - \int_c^{c+\delta} v\alpha(v)(1-\beta(v)+\overline{\mu}\beta(v))dG(v) - \lambda \mathbb{E}_s[\widetilde{m}_A(s) - m_A(s) | s > \hat{s}_A](1-\hat{s}_A) \\
&\geq \int_{1-\varepsilon}^1 v dG(v) - \int_c^{c+\delta} v dG(v) - \lambda \int_{\hat{s}_A}^1 \Big(\int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds \\
&= -\lambda \int_{\hat{s}_A}^1 \Big(\int_{1-\varepsilon}^1 1 dG(v) - \int_c^{c+\delta} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds
\end{aligned}
$$

$$
\begin{aligned}
&> -\lambda\int_{\hat{s}_A}^{1}\Big(\int_{1-\varepsilon}^{1} 1dG(v) - \int_{c}^{c+\delta}\alpha(v)(1-\beta(v)+\mu(\hat{s}_A)\beta(v))dG(v)\Big)ds\\
&=0,
\end{aligned}
$$

where the first inequality holds since $\alpha(v), \beta(v), \overline{\mu} \leq 1$ for any v, the first equality follows from (A.1.3), and the last inequality follows from the fact that $\mu(\cdot)$ is strictly increasing in s, and the last equality follows from (A.1.4). $\square$

Step 2. $\mathcal{V}_A \cup \mathcal{V}_B$ *is a connected interval.*

Proof . Suppose that there is gap in $\mathcal{V}_A \cup \mathcal{V}_B$. The proof is analogous to Step 1, where $(1-\varepsilon, 1]$ is now replaced by the gap in $\mathcal{V}_A \cup \mathcal{V}_B$. We omit the details. $\square$

Step 3. $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$

Proof . Suppose to the contrary that $\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} = 0$. Suppose $\inf\{\mathcal{V}_A\} = 0$. Let A reject a small fraction of students at the bottom, $[0, \varepsilon)$, where $2\varepsilon < 1 - \hat{s}_A$ and $\hat{s}_A$ is such that $m_A(\hat{s}_A) = \kappa$. Then, the mass of students attending A from the deviation is

$$
\widetilde{m}_A(s) = \int_{\varepsilon}^{1}\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v).
$$

Denote $\tilde{s}_A$ be the state such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A$ since $\widetilde{m}_A(s) < m_A(s)$. Hence, we can choose ε such that $\tilde{s}_A - \hat{s}_A < \varepsilon$. Then, A's net payoff from the deviation is

$$
\widetilde{\pi}_A - \pi_A = \underbrace{-\int_{0}^{\varepsilon} v\alpha(v)(1-\beta(v)+\overline{\mu}\beta(v))dG(v)}_{(*)} + \underbrace{\lambda\left[\int_{\hat{s}_A}^{1}(m_A(s)-\kappa)ds - \int_{\tilde{s}_A}^{1}(\widetilde{m}_A(s)-\kappa)ds\right]}_{(**)}.
$$

Note that

$$
\begin{aligned}
(*) &= \int_{0}^{1}\Big(\int_{0}^{\varepsilon} v\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds\\
&< \varepsilon\left[\int_{\tilde{s}_A}^{1}\Big(\int_{0}^{\varepsilon}\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds + \int_{0}^{\tilde{s}_A}\Big(\int_{0}^{\varepsilon}\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds\right]
\end{aligned} \tag{A.1.5}
$$

and

$$
\begin{aligned}
(**) &= \int_{\tilde{s}_A}^{1}(m_A(s)-\kappa)ds + \int_{\hat{s}_A}^{\tilde{s}_A}(m_A(s)-\kappa)ds - \int_{\tilde{s}_A}^{1}(\widetilde{m}_A(s)-\kappa)ds\\
&= \int_{\tilde{s}_A}^{1}\Big(\int_{0}^{\varepsilon}\alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v)\Big)ds + \int_{\hat{s}_A}^{\tilde{s}_A}(m_A(s)-\kappa)ds
\end{aligned}
$$

$$> \int_{\tilde{s}_A}^{1} \Big(\int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds \tag{A.1.6}$$

where the last inequality holds since $m_A(s) > \kappa$ for any $s \in (\hat{s}_A, \tilde{s})$. Thus, we have

$$\begin{aligned}
&\widetilde{\pi}_A - \pi_A \\
>&(\lambda-\varepsilon) \int_{\tilde{s}_A}^{1} \Big(\int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds - \varepsilon \int_0^{\tilde{s}_A} \Big(\int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(s)\beta(v))dG(v) \Big) ds \\
>&(\lambda-\varepsilon)(1-\tilde{s}_A) \int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(\tilde{s}_A)\beta(v))dG(v) - \varepsilon\, \tilde{s}_A \int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(\tilde{s}_A)\beta(v))dG(v) \\
=&\big(\lambda(1-\tilde{s}_A)-\varepsilon\big) \int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(\tilde{s}_A)\beta(v))dG(v) \\
>&\varepsilon(\lambda-1) \int_0^{\varepsilon} \alpha(v)(1-\beta(v)+\mu(\tilde{s}_A)\beta(v))dG(v) \\
\geq&0
\end{aligned}$$

where the penultimate inequality holds since $\lambda(1-\tilde{s}_A)-\varepsilon = \lambda((1-\hat{s}_A)-(\tilde{s}_A-\hat{s}_A))-\varepsilon > \lambda\varepsilon-\varepsilon = \varepsilon(\lambda-1)$ because $\tilde{s}_A - \hat{s}_A < \varepsilon$ and $2\varepsilon < 1-\hat{s}_A$. □ ■

A.2 Non-Competitive Equilibrium

In this section, we show that when $\kappa < \frac{1}{2}$ is not too small or $\lambda > 1$ is not too large, there does not exist a non-competitive equilibrium.

Lemma A1. *Suppose that $\mathcal{V}_{AB}$ has zero measure. Then, we have the followings:*

(i) There is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, one college has an incentive to deviate.

(ii) There is $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, one college has an incentive to deviate.

Proof. Since $\mathcal{V}_{AB}$ has zero measure, $m_i(s) = \kappa$ for all s and

$$\pi_i = \int_{\mathcal{V}_i} v\, dG(v), \quad i = A, B.$$

Now, let $\underline{c}_i := \inf\{\mathcal{V}_i\}$ and $\bar{c}_i := \sup\{\mathcal{V}_i\}$.

Proof of (i) . Let $\underline{c}_A = \inf\{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\underline{c}_A = G^{-1}(1-2\kappa)$ by Lemma 1. We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A+\delta]$ but accepts those in $[\bar{c}_B-\varepsilon, \bar{c}_B]$, where ε and δ are such that

$$G(\bar{c}_B) - G(\bar{c}_B-\varepsilon) = G(\underline{c}_A+\delta) - G(\underline{c}_A). \tag{A.2.1}$$

Note that the mass of students attending A under this deviation is

$$\begin{aligned}
\widetilde{m}_A(s) &= \int_{\overline{c}_B-\varepsilon}^{\overline{c}_B} \mu(s)\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} 1\, dG(v) \\
&= \mu(s)[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon)] + \kappa - [G(\underline{c}_A + \delta) - G(\underline{c}_A)] \\
&\leq \kappa,
\end{aligned}$$

where the second equality holds since $m_A(s) = \kappa$ for all s, and the last inequality follows from (A.2.1) and the fact that $\mu(s) \leq 1$ for all s.

Since A is never over-demanded, its payoff from the deviation is

$$\widetilde{\pi}_A = \overline{\mu} \int_{\overline{c}_B-\varepsilon}^{\overline{c}_B} v\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} v\, dG(v) = \overline{\mu} \int_{\overline{c}_B-\varepsilon}^{\overline{c}_B} v\, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v).$$

Therefore,

$$\begin{aligned}
&\tilde{\pi}_A - \pi_A && \text{(A.2.2)} \\
=&\overline{\mu} \int_{\overline{c}_B-\varepsilon}^{\overline{c}_B} v\, dG(v) - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v) \\
=&\overline{\mu} \left[\overline{c}_B G(\overline{c}_B) - (\overline{c}_B - \varepsilon) G(\overline{c}_B - \varepsilon) - \int_{\overline{c}_B-\varepsilon}^{\overline{c}_B} G(v) dv\right] - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \int_{\underline{c}_A}^{\underline{c}_A+\delta} G(v) dv\right] \\
>&\overline{\mu} \left[\overline{c}_B G(\overline{c}_B) - (\overline{c}_B - \varepsilon) G(\overline{c}_B - \varepsilon) - \varepsilon G(\overline{c}_B)\right] - \left[(\underline{c}_A + \delta) G(\underline{c}_A + \delta) - \underline{c}_A G(\underline{c}_A) - \delta G(\underline{c}_A)\right] \\
=&\big[G(\overline{c}_B) - G(\overline{c}_B - \varepsilon)\big] \big[\overline{\mu}\, \overline{c}_B - \underline{c}_A - \overline{\mu}\, \varepsilon - \delta\big], && \text{(A.2.3)}
\end{aligned}$$

where the first equality follows from the integration by parts, and the last equality follows from (A.2.1). Observe that if $\overline{\mu} > \frac{\underline{c}_A}{\overline{c}_B}$, then (A.2.2) is strictly positive for sufficiently small ε and δ, hence $\tilde{\pi}_A > \pi_A$. Note that since $\underline{c}_A = G^{-1}(1 - 2\kappa)$ and $m_i(s) = \kappa$ for all s and $i = A, B$, we have that $G(\overline{c}_B) \geq 1 - \kappa$; that is, $\overline{c}_B \geq G^{-1}(1 - \kappa)$. (Otherwise, college A must be admitting more than measure κ of students.) Therefore,

$$\frac{\underline{c}_A}{\overline{c}_B} \leq \frac{G^{-1}(1 - 2\kappa)}{G^{-1}(1 - \kappa)}. \tag{A.2.4}$$

Since the RHS of (A.2.4) is continuous in κ and converges to zero as κ approaches to $\frac{1}{2}$, there is $\hat{\kappa} < \frac{1}{2}$ such that for any $\kappa > \hat{\kappa}$, $\overline{\mu} > \frac{\underline{c}_A}{\overline{c}_B}$ for any given $\overline{\mu}$. □

Proof of (ii) . Let $\overline{c}_B = \sup\{\mathcal{V}_A \cup \mathcal{V}_B\}$, without loss of generality. Then, $\overline{c}_B = 1$ by Lemma 1. We show that college A has an incentive to deviate. Suppose A rejects students in $[\underline{c}_A, \underline{c}_A + \delta]$ but admits students in $[1-\varepsilon, 1]$, where ε and δ satisfy

$$\mu(1-\underline{c}_A)[1-G(1-\varepsilon)] = G(\underline{c}_A+\delta) - G(\underline{c}_A). \tag{A.2.5}$$

The mass of students attending A in state s under the deviation is

$$\widetilde{m}_A(s) = \int_{1-\varepsilon}^{1} \mu(s)\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} 1\, dG(v) = \mu(s)[1-G(1-\varepsilon)] + \kappa - [G(\underline{c}_A+\delta) - G(\underline{c}_A)].$$

Let $\hat{s}_A$ be such that $\widetilde{m}_A(\hat{s}_A) = \kappa$, i.e., $\mu(\hat{s}_A)[1-G(1-\varepsilon)] = [G(\underline{c}_A+\delta) - G(\underline{c}_A)]$. Since $\mu(\cdot)$ is strictly increasing in s, $\hat{s}_A = 1-\underline{c}_A$ by (A.2.5).

Thus, A's payoff from the deviation is

$$\begin{aligned}
\widetilde{\pi}_A &= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) + \int_{\mathcal{V}_A \setminus [\underline{c}_A, \underline{c}_A+\delta]} v\, dG(v) - \lambda \int_{\hat{s}_A}^{1} [m(s) - \kappa]\, ds \\
&= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) + \pi_A - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v) \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big].
\end{aligned}$$

and the net payoff from the deviation is

$$\begin{aligned}
\tilde{\pi}_A - \pi_A &= \overline{\mu} \int_{1-\varepsilon}^{1} v\, dG(v) - \int_{\underline{c}_A}^{\underline{c}_A+\delta} v\, dG(v) \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big] \\
&> \overline{\mu}(1-\varepsilon)[1-G(1-\varepsilon)] - (\underline{c}_A+\delta)[G(\underline{c}_A+\delta) - G(\underline{c}_A)] \\
&\quad - \lambda \Big[(1-G(1-\varepsilon)) \int_{\hat{s}_A}^{1} \mu(s) ds - [G(\underline{c}_A+\delta) - G(\underline{c}_A)](1-\hat{s}_A) \Big] \\
&= [1-G(1-\varepsilon)] \Big(\overline{\mu} - \eta\, \underline{c}_A - \overline{\mu}\, \varepsilon - \eta\, \delta - \lambda \Big[\int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1-\hat{s}_A) \Big] \Big),
\end{aligned} \tag{A.2.6}$$

where $\eta = \mu(1-\underline{c}_A)$ and the last equality follows from (A.2.5).

Observe that if $\overline{\mu} - \eta\, \underline{c}_A - \lambda \Big[\int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1-\hat{s}_A) \Big] > 0$, then (A.2.6) is strictly positive for sufficiently small ε and δ. Note that

$$\overline{\mu} - \eta\, \underline{c}_A - \lambda \Big[\int_{\hat{s}_A}^{1} \mu(s) ds - \eta(1-\hat{s}_A) \Big] = \overline{\mu} - \lambda \int_{\hat{s}_A}^{1} \mu(s)\, ds + (\lambda - 1)\, \eta\, \underline{c}_A,$$

Since $\overline{\mu} = \int_0^1 \mu(s)\, ds > \int_{\hat{s}_A}^1 \mu(s)\, ds$ (which follows from the fact that $\hat{s}_A < 1$), there exists $\hat{\lambda} > 1$ such that for any $\lambda < \hat{\lambda}$, $\tilde{\pi}_A > \pi_A$. $\square$ ■

A.3 Proof of Lemma 2

Observe first that $H_i(\cdot, x)$ is strictly increasing in v, since for $v' > v$,

$$H_A(v', x) - H_A(v, x) = (1 - x + \overline{\mu}x)(v' - v) > 0$$

and similar for H_B.

Next, we show that there is an interior cutoff $\hat{\sigma}_A$ such that $H_A(v, x)$ satisfies the single crossing property with respect to x; that is, if $H_A(v, x) \leq 0$ for some $x \in (0, 1)$, then $H_A(v, x') < 0$ for any $x' > x$. Suppose for any $x \in (0, 1)$,

$$H_A(v, x) = (1 - x)[v - \lambda(1 - \hat{s}_A)] + x\overline{\mu}\big[v - \lambda(1 - \hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}\big] \leq 0. \tag{A.3.1}$$

Consider any $x' > x$. If $v < \lambda(1 - \hat{s}_A)$, then

$$H_A(v, x') = (1 - x')[v - \lambda(1 - \hat{s}_A)] + x'\overline{\mu}\big[v - \lambda(1 - \hat{s}_A)\frac{\mu_+(\hat{s}_A)}{\overline{\mu}}\big] < 0,$$

where the inequality follows from (A.3.1) and the facts that $x' > x$ and $\mu_+(\hat{s}_A) > \overline{\mu}$. If $v \geq \lambda(1-\hat{s}_A)$, then

$$\begin{aligned} H_A(v, x) - H_A(v, x') &= (x' - x)\big[v(1 - \overline{\mu}) - \lambda(1 - \hat{s}_A)(1 - \mu_+(\hat{s}_A))\big] \\ &> (x' - x)[v - \lambda(1 - \hat{s}_A)](1 - \mu_+(\hat{s}_A)) \\ &\geq 0, \end{aligned}$$

where the first inequality holds since $x' > x$ and $\mu_+(\hat{s}_A) > \overline{\mu}$, and the second inequality holds since $v \geq \lambda(1 - \hat{s}_A)$. Since $H_A(v, x) \leq 0$, we thus have $H_A(v, x') < 0$. The proof for H_B is analogous.

A.4 Proofs of Theorem 2

Suppose to the contrary that $\hat{v} \leq \check{v}$ in a competitive equilibrium. Suppose further that

$$\underline{v}_B < \overline{v}_B \leq \underline{v}_A < \overline{v}_A, \tag{A.4.1}$$

without loss of generality, where the first and the last strict inequalities hold since $(\hat{s}_A, \hat{s}_B) \in (0, 1)^2$ by Lemma 1-(iii). Note that we must have $\overline{v}_A \in (0, 1)$ in equilibrium, since if $\overline{v}_A = 1$, then $m_A(s) = 0$ for all s, and if $\overline{v}_A = 0$, then $\underline{v}_B = \overline{v}_B = \underline{v}_A = \overline{v}_A = 0$, so $m_B(s) = 0$ for all s. In equilibrium, we have

$$m_A(\hat{s}_A) = \mu(\hat{s}_A)[1 - G(\overline{v}_A)] = \kappa, \tag{A.4.2}$$

and

$$m_B(\hat{s}_B) = (1-\mu(\hat{s}_B))[1-G(\overline{v}_A)] + G(\overline{v}_A) - G(\underline{v}_B) = \kappa. \tag{A.4.3}$$

From (A.4.2), $1 - G(\overline{v}_A) = \frac{\kappa}{\mu(\hat{s}_A)}$. Substituting this into (A.4.3), we have

$$G(\overline{v}_A) - G(\underline{v}_B) = \kappa\left(\frac{\mu(\hat{s}_A)+\mu(\hat{s}_B)-1}{\mu(\hat{s}_A)}\right).$$

Since $\underline{v}_B < \overline{v}_A$, this implies

$$\mu(\hat{s}_A) + \mu(\hat{s}_B) > 1 \Leftrightarrow \mu(\hat{s}_B) > 1 - \mu(\hat{s}_A) = \mu(1-\hat{s}_A),$$

where the last equality follows from the symmetry of $\mu(\cdot)$. Since $\mu(\cdot)$ is strictly increasing, we have $\hat{s}_B > 1 - \hat{s}_A$, and so $\underline{v}_B = \lambda\,\hat{s}_B > \lambda\,(1-\hat{s}_A) = \underline{v}_A$ which contradicts (A.4.1).

A.5 Proof of Theorem 3

Step 1: Existence of a profile of admission strategies (α, β) that forms local best responses.

We first establish existence of a profile of admission strategies $(\alpha,\beta) : [0,1]^2 \to [0,1]^2$ such that for each $v \in [0,1]$,

$$\alpha(v;\hat{s}) = \begin{cases} 1 & \text{if } H_A(v,1;\hat{s}) > 0 \\ 0 & \text{if } H_A(v,1;\hat{s}) < 0,\ \ H_B(v,1;\hat{s}) > 0 \\ \alpha_0(v;\hat{s}) & \text{if } H_A(v,1;\hat{s}) < 0 < H_A(v,0;\hat{s}),\ \ H_B(v,1;\hat{s}) < 0 < H_B(v,0;\hat{s}) \\ 1 & \text{if } H_A(v,0;\hat{s}) > 0,\ \ H_B(v,0;\hat{s}) < 0 \\ 0 & \text{if } H_A(v,0;\hat{s}) < 0 \end{cases} \tag{A.5.1}$$

and

$$\beta(v;\hat{s}) = \begin{cases} 1 & \text{if } H_B(v,1;\hat{s}) > 0 \\ 0 & \text{if } H_B(v,1;\hat{s}) < 0,\ \ H_A(v,1) > 0 \\ \beta_0(v;\hat{s}) & \text{if } H_B(v,1;\hat{s}) < 0 < H_B(v,0;\hat{s}),\ \ H_A(v,1;\hat{s}) < 0 < H_A(v,0;\hat{s}) \\ 1 & \text{if } H_B(v,0;\hat{s}) > 0,\ \ H_A(v,0;\hat{s}) < 0 \\ 0 & \text{if } H_B(v,0;\hat{s}) < 0 \end{cases}, \tag{A.5.2}$$

where $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ are respectively given by (3.2) and (3.3), and $\hat{s} = (\hat{s}_A, \hat{s}_B)$ is given by (3.4).[22]

[22]One can also structure the strategy profile to satisfy the requirements of an A-priority equilibrium by replacing $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ with 1 and 0, respectively, and of a B-priority equilibrium by replacing them with 0 and 1, respectively.

Now, fix any $\hat{s} \in S \equiv [0,1]^2$ and consider the resulting profile $(\alpha(\cdot;\hat{s}), \beta(\cdot;\hat{s}))$. This strategy profile in turn induces the mass of students enrolling in colleges A and B: For college A,

$$m_A(s;\hat{s}) = \int_0^1 \alpha(v;\hat{s})[1 - \beta(v;\hat{s}) + \mu(s)\beta(v;\hat{s})]\, dG(v)$$

and similarly for college B,

$$m_B(s;\hat{s}) = \int_0^1 \beta(v;\hat{s})[1 - \alpha(v;\hat{s}) + (1-\mu(s))\alpha(v;\hat{s})]\, dG(v).$$

Observe that $m_A(\cdot;\hat{s})$ and $m_B(\cdot;\hat{s})$ in turn yield a new profile of cutoff states:

$$\tilde{s}_A = \inf\left\{s \in [0,1] | m_A(s;\hat{s}) - \kappa > 0\right\}, \tag{A.5.3}$$

if the set in the RHS is nonempty, or else $\tilde{s}_A \equiv 1$, and

$$\tilde{s}_B = \sup\left\{s \in [0,1] | m_B(s;\hat{s}) - \kappa > 0\right\}, \tag{A.5.4}$$

if the set in the RHS is nonempty, or else $\tilde{s}_B \equiv 0$.

Next, define a mapping T such that $T(\hat{s}) = \tilde{s}$, where $\tilde{s} = (\tilde{s}_A, \tilde{s}_B)$ is given by (A.5.3) and (A.5.4). The next lemma shows that T is continuous. Therefore, it has a fixed point by the Brouwer's fixed point theorem. From the construction of T, it is immediate that given the fixed point, say $\hat{s}^*$, the profile $(\alpha(\cdot;\hat{s}^*), \beta(\cdot;\hat{s}^*))$ satisfies the local incentives.

Lemma A2. *$T(\cdot)$ is continuous in s for $s \in S$.*

Proof. Notice, first, that $\overline{v}_A$ and $\underline{v}_A$ are continuous in $\hat{s}_A$, and $\overline{v}_B$ and $\underline{v}_B$ are continuous in $\hat{s}_B$. Now, let

$$\underline{v} := \min\left\{\underline{v}_A, \underline{v}_B\right\}, \quad \check{v} := \max\left\{\underline{v}_A, \underline{v}_B\right\}, \quad \hat{v} := \min\left\{\overline{v}_A, \overline{v}_B\right\}, \quad \overline{v} := \max\left\{\overline{v}_A, \overline{v}_B\right\}.$$

For any given $\hat{s}$, $T(\hat{s}) = \tilde{s}$ is given by (A.5.3) and (A.5.4). Consider now any $\hat{s}' = (\hat{s}'_A, \hat{s}'_B) \in S$, where $\hat{s}' \neq \hat{s}$. Then, $\alpha' \equiv \alpha(\cdot;\hat{s}')$ and $\beta' \equiv \beta(\cdot;\hat{s}')$ are defined by (A.5.1) and (A.5.2). Let

$$\underline{v}' := \min\left\{\underline{v}'_A, \underline{v}'_B\right\}, \quad \check{v}' := \max\left\{\underline{v}'_A, \underline{v}'_B\right\}, \quad \hat{v}' := \min\left\{\overline{v}'_A, \overline{v}'_B\right\}, \quad \overline{v}' := \max\left\{\overline{v}'_A, \overline{v}'_B\right\}.$$

Again, $\tilde{s}' = (\tilde{s}'_A, \tilde{s}'_B) \in S$ is defined by T through (A.5.3) and (A.5.4).

Next, let

$$v_1 := \min\left\{\underline{v}, \underline{v}'\right\}, \quad v_2 := \max\left\{\underline{v}, \underline{v}'\right\}, \quad v_3 := \min\left\{\check{v}, \check{v}'\right\}, \quad v_4 := \max\left\{\check{v}, \check{v}'\right\},$$

$$v_5 := \min\left\{\hat{v}, \hat{v}'\right\}, \quad v_6 := \max\left\{\hat{v}, \hat{v}'\right\}, \quad v_7 := \min\left\{\underline{v}, \overline{v}'\right\}, \quad v_8 := \max\left\{\overline{v}, \overline{v}'\right\},$$

and consider a partition of $[0,1]$ such that

$$\mathcal{V}_1 = (\cup_{i=2,4,6,8}[v_{i-1}, v_i]) \cap [0,1], \quad \mathcal{V}_2 = [v_4, v_5] \cap [0,1], \quad \mathcal{V}_3 = [0,1] \setminus (\mathcal{V}_1 \cup \mathcal{V}_2).$$

Consider α and α'. For any $v \in [0,1]$, we have

$$\int_0^1 \left|\alpha'(v) - \alpha(v)\right| \, dG(v) = \sum_{i=1}^{3} \int_0^1 \left|\alpha'(v) - \alpha(v)\right| \mathbb{1}_{\mathcal{V}_i}(v) \, dG(v),$$

where $\mathbb{1}_{\mathcal{V}_i}(v)$ is 1 if $v \in \mathcal{V}_i$ or 0 otherwise.

Observe, first, that by the continuity of $\underline{v}_i$ and $\overline{v}_i$, $i = A, B$, there is a $\delta_1 > 0$ such that for any $\varepsilon > 0$, if $\|\hat{s}' - \hat{s}\| < \delta_1$, then

$$\int_0^1 \mathbb{1}_{\mathcal{V}_1}(v) \, dG(v) < \frac{\varepsilon}{6}. \tag{A.5.5}$$

Second, for any $v \in \mathcal{V}_2$, the continuity of $\alpha_0(\cdot)$, given by (3.2), implies that there is δ_2 such that $\|\hat{s}' - \hat{s}\| < \delta_2$ implies

$$\left|\alpha'(v) - \alpha(v)\right| = \left|\alpha_0'(v) - \alpha_0(v)\right| < \frac{\varepsilon}{6}, \tag{A.5.6}$$

Lastly, for any $v \in \mathcal{V}_3$, $\alpha'(v)$ and $\alpha(v)$ are either 0 or 1 at the same time, hence we have that

$$\left|\alpha(v') - \alpha(v)\right| = 0. \tag{A.5.7}$$

Now, let $\delta := \min\{\delta_1, \delta_2\}$ and suppose $\|\hat{s}' - \hat{s}\| < \delta$. Then, we have

$$\begin{aligned}\int_0^1 \left|\alpha'(v) - \alpha(v)\right| \, dG(v) &= \int_0^1 \left|\alpha'(v) - \alpha(v)\right| \mathbb{1}_{\mathcal{V}_1}(v) \, dG(v) + \int_0^1 \left|\alpha_o'(v) - \alpha_o(v)\right| \mathbb{1}_{\mathcal{V}_2}(v) \, dG(v) \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3},\end{aligned} \tag{A.5.8}$$

where the equality follows from (A.5.7) and the inequality follows from (A.5.5) and (A.5.6).

Similarly, we also have that

$$\int_0^1 \left|\beta'(v) - \beta(v)\right| dG(v) < \frac{\varepsilon}{3}. \tag{A.5.9}$$

Observe that

$$\begin{aligned}&\left|\int_0^1 \alpha'(v)[1 - \beta'(v) + \mu(s)\beta'(v)] \, dG(v) - \int_0^1 \alpha(v)[1 - \beta(v) + \mu(s)\beta(v)] \, dG(v)\right| \\ &= \left|\int_0^1 \Big[[\alpha'(v) - \alpha(v)] - (1 - \mu(s))[\alpha'(v)\beta'(v) - \alpha(v)\beta(v)]\Big] \, dG(v)\right| \\ &\leq \int_0^1 \left|\alpha'(v) - \alpha(v)\right| \, dG(v) + (1 - \mu(s)) \int_0^1 \left|\alpha'(v)\beta'(v) - \alpha(v)\beta(v)\right| \, dG(v)\end{aligned} \tag{A.5.10}$$

The first part of (A.5.10) is smaller than $\varepsilon/3$ by (A.5.8). The second part of (A.5.10) is

$$\begin{aligned}\int_0^1 \left|\alpha(v)\beta'(v)-\alpha(v)\beta(v)\right| dG(v) &= \int_0^1 \left|a'(v)\beta'(v)-\alpha'(v)\beta(v)+\alpha'(v)\beta(v)-\alpha(v)\beta(v)\right| dG(v)\\ &\leq \int_0^1 \left|\beta'(v)-\beta(v)\right| dG(v)+\int_0^1 \left|\alpha'(v)-\alpha(v)\right| dG(v)\\ &< \frac{2}{3}\varepsilon,\end{aligned}$$

where the first inequality holds since $\alpha'(v), \beta(v) \leq 1$, and the last inequality follows from (A.5.8) and (A.5.9). Therefore, if $\|\hat{s}'-\hat{s}\|<\delta$, then

$$\left|\int_0^1 \alpha'(v)[1-\beta'(v)+\mu(s)\beta'(v)]dG(v)-\int_0^1 \alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v)\right|<\varepsilon. \tag{A.5.11}$$

Similarly, we also have

$$\left|\int_0^1 \beta'(v)[1-\alpha'(v)+(1-\mu(s))\alpha'(v)]dG(v)-\int_0^1 \beta(v)[1-\alpha(v)+(1-\mu(s))\alpha(v)]dG(v)\right|<\varepsilon. \tag{A.5.12}$$

Combining (A.5.11) and (A.5.12), we conclude that there is $\delta>0$ such that for any $\varepsilon>0$, if $\|\hat{s}'-\hat{s}\|<\delta$, then $\|\tilde{s}'-\tilde{s}\|<\varepsilon$. Since $\hat{s}$ is chosen arbitrary, T is continuous on S. ■

Step 2: $\mathcal{V}_{AB}$ has a positive measure in the strategy profile identified in Step 1.

Suppose to the contrary that $\mathcal{V}_{AB}$ has measure zero. Then, $\hat{s}_B^*=0$ and $\hat{s}_A^*=1$. But in that case, $H_A(v,1)>0$ and $H_B(v,1)>0$ for all v. Hence, $\overline{v}_A=\overline{v}_B=0$. Therefore, we cannot have a non-competitive equilibrium.

Step 3: The identified strategies are mutual (global) best responses.

Here, we focus on college A, since the proof for college B is analogous. We first show that π_A is concave in α and then show that $V(\cdot)$ is concave in t.

Lemma A3. *π_A is concave in α.*

Proof. Recall that

$$\begin{aligned}\pi_A = &\int_0^1 \Big[\int_0^1 v\alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v)\Big]ds\\ &-\lambda\int_0^1 \max\left\{\int_0^1 \alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v)-\kappa, 0\right\}ds.\end{aligned} \tag{A.5.13}$$

Consider any feasible α and α'. Note that for $\eta \in [0,1]$, the first part of (A.5.13) is linear in α, since

$$\int_0^1 v[\eta\alpha(v) + (1-\eta)\alpha'(v)][1-\beta(v)+\mu(s)\beta(v)]dG(v)$$
$$= \eta\int_0^1 v\alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v) + (1-\eta)\int_0^1 v\alpha'(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v),$$

and the second part is convex in α, since

$$\begin{aligned}
&\max\left\{\int_0^1 [\eta\alpha(v) + (1-\eta)\alpha'(v)][1-\beta(v)+\mu(s)\beta(v)]dG(v) - \kappa, 0\right\}\\
&= \max\left\{\eta\Big[\int_0^1 \alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v) - \kappa\Big]\right.\\
&\quad \left. + (1-\eta)\Big[\int_0^1 \alpha'(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v) - \kappa\Big], 0\right\}\\
&\leq \eta\max\left\{\int_0^1 \alpha(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v) - \kappa, 0\right\}\\
&\quad + (1-\eta)\max\left\{\int_0^1 \alpha'(v)[1-\beta(v)+\mu(s)\beta(v)]dG(v) - \kappa, 0\right\}.
\end{aligned}$$

Therefore, we have $\pi_A(\eta\alpha + (1-\eta)\alpha') \geq \eta\,\pi_A(\alpha) + (1-\eta)\pi_A(\alpha')$. ■

Lemma A4. *$V(\cdot)$ is concave in t for any $t \in [0,1]$.*

Proof. Observe that $\alpha(v;t)$, given by (3.5), is linear in t, since for any $t, t' \in [0,1]$,

$$\begin{aligned}
\alpha(v;\eta\,t + (1-\eta)\,t') &= (\eta\,t + (1-\eta)\,t')\tilde{\alpha}(v) + (1-(\eta\,t + (1-\eta)\,t'))\alpha(v)\\
&= [\eta\,t\,\tilde{\alpha}(v) + \eta\,(1-t)\,\alpha(v)] + [(1-\eta)\,t'\tilde{\alpha}(v) + (1-\eta)\,(1-t)\,\alpha(v)]\\
&= \eta\,\alpha(v;t) + (1-\eta)\,\alpha(v;t'). \qquad \text{(A.5.14)}
\end{aligned}$$

We thus have

$$\begin{aligned}
V(\eta\,t + (1-\eta)\,t') = \pi_A(\alpha(v;\eta\,t + (1-\eta)\,t')) &= \pi_A(\eta\,\alpha(v;t) + (1-\eta)\,\alpha(v;t'))\\
&\geq \eta\,\pi_A(\alpha(v;t)) + (1-\eta)\,\pi_A(\alpha(v;t'))\\
&= \eta V(t) + (1-\eta)\,V(t'),
\end{aligned}$$

where the second equality follows from Lemma A3 and the inequality follows from (A.5.14). ■

We establish one more lemma showing that $V'(0) \leq 0$.

Lemma A5. $V'(0) \leq 0$.

Proof. Let

$$W(t,\hat{s}_A(t)) := \int_0^1 v\alpha(v;t)[1-\beta(v)+\overline{\mu}\beta(v)]\,dG(v) \\ -\lambda\int_{\hat{s}_A(t)}^1\Big[\int_0^1 \alpha(v;t)[1-\beta(v)+\mu(s)\beta(v)]\,dG(v)-\kappa\Big]\,ds$$

and denote it by $V(t) := W(t,\hat{s}_A(t))$. Observe that

$$V'(t) = W_1(t,\hat{s}_A(t)) + W_2(t,\hat{s}_A(t))\hat{s}'_A(t),$$

where

$$W_1(t,\hat{s}_A(t)) = \int_0^1 (\tilde{\alpha}(v)-\alpha(v))\Big[v[1-\beta(v)+\overline{\mu}\beta(v)] - \lambda\int_{\hat{s}_A(t)}^1 [1-\beta(v)+\mu(s)\beta(v)]ds\Big]\,dG(v)$$

and

$$W_2(t,\hat{s}_A(t)) = \lambda\Big[\int_0^1 \alpha(v;t)[1-\beta(v)+\mu(\hat{s}_A(t))\beta(v)]\,dG(v)-\kappa\Big].$$

Notice that $W_2(0,\hat{s}_A(0)) = 0$ by definition of $\hat{s}_A$. Therefore, we have

$$V'(0) = W_1(0,\hat{s}(0)) = \int_0^1 (\tilde{\alpha}(v)-\alpha(v))H_A(v,\beta(v))\,dG(v) \leq 0,$$

where the inequality holds since if $H_A(v,\beta(v)) \geq 0$ for some v, then $\alpha(v) = 1$ and $\tilde{\alpha}(v) \leq 1$ for such v; if $H_A(v,\beta(v) \leq 0$ for some v, then $\alpha(v) = 0$ and $\tilde{\alpha}(v) \geq 0$ for such v; and $H_A(v,\beta(v)) = 0$ otherwise. ∎

Observe that

$$\pi_A(\tilde{\alpha}) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_A(\alpha),$$

where the first inequality follows from the concavity of $V(\cdot)$ and the second inequality follows from Lemma A5. This completes the proof.

A.6 Proof of Theorem 4

Proof of Part (i). Consider any non-competitive equilibrium. For each state s except $\mu(s) = 0$ or 1, the equilibrium must admit a positive measure of students who prefer A but are assigned to B and a positive measure of students who are assigned to A but have scores lower than those of the first group of students; that is, justified envy arises. Since justified envy arises for a positive measure of students for almost every state,[23] the outcome is unfair. Also, for almost every state,

[23]since $\mu(\cdot)$ is strictly increasing and continuous in s, $\mu(s) \in (0,1)$ for almost every state.

there must be a positive measure of students assigned to A but prefer B and a positive measure of students assigned to B but prefer A. Thus, the outcome is student inefficient.

Next, the equilibrium is college efficient. To see this, recall that in any non-competitive equilibrium, almost all top 2κ students are assigned to either college. Suppose to the contrary that for a given state, there is another assignment that makes both colleges weakly better off and at least one college strictly better off. Then, it must also admit almost all top 2κ students, or else at least one college is strictly worse off. Therefore, it is a reallocation of the initial assignment, hence if one college is strictly better off, then the other college must be strictly worse off. Thus, we reach a contraction. ■

Proof of Part (ii). Suppose that almost all top κ students are assigned to one college, and the next top κ students are assigned to the other college. Then, any change of assignments by positive measure of students will leave the former college strictly worse off, hence it is Pareto efficient.

Suppose it is not the case in a non-competitive equilibrium. Note that for a fixed s, there are some $\mathcal{V}'_i, \mathcal{V}''_i \subset \mathcal{V}_i$ and $\mathcal{V}'_j \subset \mathcal{V}_j$, $i \neq j$, all with positive measures, such that $v' < \hat{v} < v''$ whenever $v' \in \mathcal{V}'_i$, $v'' \in \mathcal{V}''_i$ and $\hat{v} \in \mathcal{V}'_j$. Let $i = A$ and $j = B$ without loss of generality. We can choose $\mathcal{V}'_A$, $\mathcal{V}''_A$ and $\mathcal{V}'_B$ that satisfy

$$\frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v)}{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1\, dG(v)} = \frac{\int_{\mathcal{V}'_B} v\, dG(v)}{\int_{\mathcal{V}'_B} 1\, dG(v)} \tag{A.6.1}$$

and

$$(1 - \mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1\, dG(v) = \mu(s) \int_{\mathcal{V}'_B} 1\, dG(v). \tag{A.6.2}$$

(If either (A.6.1) or (A.6.2) is violated, we can adjust $\mathcal{V}'_A$, $\mathcal{V}''_A$ and/or $\mathcal{V}'_B$ by adding or subtracting a positive mass of students.) Note that the LHS (resp. RHS) of (A.6.2) is the measure of students who prefer B (resp. A) in $\mathcal{V}'_A \cup \mathcal{V}''_A$ (resp. $\mathcal{V}'_B$). From (A.6.1), we have

$$\begin{aligned}
&\frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v)}{(1-\mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} 1\, dG(v)} = \frac{\int_{\mathcal{V}'_B} v\, dG(v)}{(1-\mu(s)) \int_{\mathcal{V}'_B} 1\, dG(v)} \\
\Leftrightarrow\ &\frac{\int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v)}{\mu(s) \int_{\mathcal{V}'_B} 1\, dG(v)} = \frac{\int_{\mathcal{V}'_B} v\, dG(v)}{(1-\mu(s)) \int_{\mathcal{V}'_B} 1\, dG(v)} \\
\Leftrightarrow\ &(1-\mu(s)) \int_{\mathcal{V}'_A \cup \mathcal{V}''_A} v\, dG(v) = \mu(s) \int_{\mathcal{V}'_B} v\, dG(v),
\end{aligned}$$

where the first equivalence follows from (A.6.2). The last equivalence shows that the average value of students who prefer B in $\mathcal{V}'_A \cup \mathcal{V}''_A$ is the same as that of students who prefer A in $\mathcal{V}'_B$. Thus, in state s, a fraction $1 - \mu(s)$ of students in $\mathcal{V}'_A \cup \mathcal{V}''_A$ who prefer B to A can be swapped with a fraction of $\mu(s)$ of students in $\mathcal{V}'_B$ who prefer A to B. This reassignment leaves both colleges the same in welfare and makes all students weakly better off and some positive measure of students strictly better off. Since this argument holds for all s except $\mu(s) = 0$ or 1, the outcome is Pareto inefficient. ■

Proof of Part (iii). Recall that there are cutoff states $(\hat{s}_A, \hat{s}_B)$ such that colleges have a mass of unfilled seats in a positive measure of states, $[0, \hat{s}_A)$ for A and $(\hat{s}_B, 1]$ for B, despite the fact that there are unmatched and acceptable students ($\inf\{\mathcal{V}_A \cup \mathcal{V}_B\} > 0$ in Lemma 1-(ii)). By assigning those unmatched students to a college with excess capacity, both the students and college are better off. Thus, it is student, college and Pareto inefficient. ■

Proof of Part (iv). Suppose a competitive equilibrium exhibits strategic targeting; i.e., $\check{v} < \hat{v}$. Fix a state s such that $\mu(s) \neq 0, 1$. For those students in $[\check{v}, \hat{v}]$, there is a positive measure of students who are assigned to a college, say B, but prefer A, and their scores are higher than those of a positive measure of students who are assigned to A, even though both colleges prefer the high-score students. Moreover, students in $[\check{v}, \hat{v}]$ get no admission from either college with positive probabilities even when their scores are high. Thus, it entails justified envy for a positive measure of states for almost every state.

Suppose now a competitive equilibrium does not exhibit strategic targeting; i.e., $\hat{v} < \check{v}$. Let $\overline{v}_B < \underline{v}_A$, as depicted in Figure 3.4, without loss of generality, so students in $[\underline{v}_B, \overline{v}_A]$ admitted only by B and those in $(\overline{v}_A, 1]$ are admitted by both colleges. Observe that only the students who are not admitted by either college or admitted only by college B may have envies. However, the students whom they envy have higher scores. So, no justified envy arises in any state s, making the outcome fair. ■

A.7 Proofs of Theorem 5 and the Existence of Cutoff Equilibrium

A.7.1 Proof of Theorem 5

Suppose there is a cutoff equilibrium with strategy profiles (α, β) where $\alpha(v, e) = \mathbb{1}_{\{e \geq \eta(v)\}}$ and $\beta(v, e') = \mathbb{1}_{\{e' \geq \xi(v)\}}$, for some $\eta(\cdot)$ and $\xi(\cdot)$ which are nonincreasing.

Here, we focus on college A, since college B's behavior is analogous. Since $U_e > 0$, by the Implicit Function Theorem, $H_A(v, e, \overline{\beta}(v)) = 0$ implicitly defines $\eta(v)$. Since $\mu_+(\hat{s}_A) > \overline{\mu}$, we must have

$$1 - \overline{\beta}(v) + \mu_+(\hat{s}_A)\overline{\beta}(v) > 1 - \overline{\beta}(v) + \overline{\mu}\overline{\beta}(v).$$

Then, $H_A(v, \eta(v), \overline{\beta}(v)) = 0$ implies that by (4.1),

$$U(v, \eta(v)) > \lambda(1 - \hat{s}_A). \tag{A.7.1}$$

Next, totally differentiate H_A to obtain:

$$\begin{aligned}
&U_v(v,\eta(v)) + U_e(v,\eta(v))\eta'(v) \\
&= \frac{1}{1-\overline{\beta}(v)+\overline{\mu}\overline{\beta}(v)}\big(U(v,\eta(v))(1-\overline{\mu}) - \lambda(1-\hat{s}_A)(1-\mu_+(\hat{s}_A))\big)\overline{\beta}'(v).
\end{aligned} \tag{A.7.2}$$

Since college B adopts a cutoff strategy, $\overline{\beta}(v) = 1 - Y(\xi(v)|v)$, we have that

$$\overline{\beta}'(v) = -y(\xi(v)|v)\xi'(v) - Y_v(\xi(v)|v) > 0, \tag{A.7.3}$$

where the inequality holds since $\xi'(v) \leq 0$ and $Y_v(e|v) < 0$.[24]

Further, $\overline{\mu} < \mu_+(\hat{s}_A) \leq 1$, so it follows from (A.7.1) that the RHS of (A.7.2) is strictly positive for any v such that $\eta(v) \in (0,1)$. Hence, for all v,

$$-\eta'(v) \leq \frac{U_v(v,\eta(v))}{U_e(v,\eta(v))}, \tag{A.7.4}$$

and the inequality is strict for a positive measure of v.

A.7.2 Existence of Cutoff Equilibrium

Step 1: Existence of a profile of cutoff strategies for A and B.

Define

$$\delta := \max_{v,e,e'}\left\{x(e|v)\left(\frac{U_v(v,e)}{U_e(v,e)}\right) - X_v(e|v) \bigvee y(e'|v)\left(\frac{V_v(v,e')}{V_{e'}(v,e')}\right) - Y_v(e|v)\right\}.$$

Let $\mathcal{M}$ be the set of Lipschitz-continuous nondecreasing functions mapping from $[0,1]$ to $[0,1]$ with Lipschitz bound given by δ.

We define an operator $T : [0,1]^2 \times \mathcal{M}^2 \to [0,1]^2 \times \mathcal{M}^2$ as follows. For any $(\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta}) \in [0,1]^2 \times \mathcal{M}^2$, the third component of $T(\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta})$ is a function $\overline{a}$ defined as follows. First, $\eta(v)$ is implicitly defined via $H_A(v,\eta(v),\overline{\beta}(v)) = 0$ according to the Implicit Function Theorem (since $U_e > 0$). For v such that $\eta(v) \in (0,1)$, the same argument as in the proof of Theorem 5 implies that

$$0 \leq -\eta'(v) \leq \frac{U_v(v,\eta(v))}{U_e(v,\eta(v))}.$$

Hence, $\eta^{-1}((0,1))$ forms an interval. For $v \leq \inf \eta^{-1}((0,1))$, we extend η such that $\eta(v) = 1$ and for

[24] When v and e are independent, $\overline{\beta}'(v) = -y(\xi(v))\xi'(v) \geq 0$. This implies that each college under-weights a students' common performance and over-weights her non-common performance at least weakly and one college does so strictly. Further, together with college B's condition (total differentiation of H_B), one can show that $\overline{\beta}'(v) > 0$ for a positive measure of v, generically.

$v \geq \sup \eta^{-1}((0,1))$, we set $\eta(v) = 0$. We now define $\alpha(v,e) := \mathbb{1}_{\{e \geq \eta(v)\}}$. Let $a(v) = \mathbb{E}_e[\alpha(v,e)|v]$. Then,

$$\overline{a}(v) = 1 - X(\eta(v)|v).$$

Since η is nonincreasing, a is nondecreasing. Further,

$$\overline{a}'(v) = -x(\eta(v)|v) - X_v(\eta(v)|v)\eta'(v) \leq x(\eta(v)|v)\frac{U_v(v,\eta(v))}{U_e(v,\eta(v))} - X_v(\eta(v)|v) \leq \delta.$$

It thus follows that $\overline{a} \in \mathcal{M}$.

The fourth component of $T(\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta})$, labeled $\overline{b}$, is analogously constructed via $e' = \xi(v)$ determined implicitly by $H_B(v, \xi(v), \overline{\alpha}) = 0$, analogously, and belongs to $\mathcal{M}$. This process also determines B's strategy β.

The first two components $(\hat{s}'_A, \hat{s}'_B)$ are determined by the $m_A(\hat{s}'_A) = m_B(\hat{s}'_B) = \kappa$, much as in the earlier proofs, using α and β, along with $(\hat{s}_A, \hat{s}_B)$ as input.

In sum, the operator T maps from $(\hat{s}_A, \hat{s}_B, \overline{\alpha}, \overline{\beta}) \in [0,1]^2 \times \mathcal{M}^2$ to $(\hat{s}'_A, \hat{s}'_B, \overline{a}, \overline{b}) \in [0,1]^2 \times \mathcal{M}^2$. By Arzela-Ascoli theorem, the set $\mathcal{M}$ endowed with sup norm topology is compact, bounded and convex. Hence, the same holds for the Cartesian product $[0,1]^2 \times \mathcal{M}^2$. Following the techniques used in Appendix B, the mapping T is continuous (with respect to sup norm). Hence, by the Schauder's theorem, T has a fixed point. The fixed point then identifies a profile of cutoff strategies (α, β) via $\alpha(v,e) = \mathbb{1}_{\{e \geq \eta(v)\}}$ and $\beta(v,e') = \mathbb{1}_{\{e' \geq \xi(v)\}}$. See Appendix B for technical details.

Step 2: The cutoff strategies identified in Step 1 form an equilibrium under a condition.

Consider the following condition:

$$\left(\frac{U_v(v,e)}{U(v,e)} + Y_v(\xi(v)|v)\right)\frac{V_{e'}(v,\tilde{e})}{V_v(v,\tilde{e})} \geq y(\tilde{e}|v)\left(\frac{\mu_+(s) - \overline{\mu}}{\mu_+(s)\overline{\mu}}\right), \forall v, e, \tilde{e}, s,$$

and

$$\left(\frac{V_v(v,e')}{V(v,e')} + X_v(\eta(v)|v)\right)\frac{U_e(v,\tilde{e})}{U_v(v,\tilde{e})} \geq x(\tilde{e}|v)\left(\frac{\overline{\mu} - \mu_-(s)}{(1-\mu_-(s))(1-\overline{\mu})}\right), \forall v, e', \tilde{e}, s,$$

where $\mu_+(s) := \mathbb{E}[\mu(\tilde{s})|\tilde{s} \geq s]$ and $\mu_-(s) := \mathbb{E}[\mu(\tilde{s})|\tilde{s} \leq s]$.

Since the RHS of each inequality is bounded by some constant, the conditions can be interpreted as requiring that each college values the non-common performance sufficiently highly. For instance, if $U(v,e) = (1-\rho)v + \rho e$ and $V(v,e') = (1-\rho)v + \rho e'$, then the LHS of each inequality will be no less than $\rho - \gamma$, where $\gamma := \max_{v,e,e'} \{|X_v(e|v)|, |Y_v(e'|v)|\}$. So the condition will hold if the RHS is less than $\rho - \gamma$.

We now show the cutoff strategies identified by Step 1 form an equilibrium, given this condition. We show this only for college A, since the argument for college B is completely analogous. For the proof, note first that $H_A(v,e,\overline{\beta}(v))$ is nondecreasing, so it suffices to show that

$$\frac{\partial H_A(v,e,\overline{\beta}(v))}{\partial v} \geq 0 \text{ whenever } H_A(v,e,\overline{\beta}(v)) = 0.$$

This result holds since

$$\begin{aligned}
&\operatorname{sgn}\left(\frac{\partial H_A(v,e,\overline{\beta}(v))}{\partial v}\right)\\
=&U_v(v,e) - \frac{(U(v,e)(1-\overline{\mu}) - \lambda(1-\hat{s}_A)(1-\mu_+(\hat{s}_A))}{1-\overline{\beta}(v)+\overline{\mu}\overline{\beta}(v)}\overline{\beta}'(v)\\
=&U_v(v,e) - U(v,e)\frac{1}{1-\overline{\beta}(v)+\overline{\mu}\overline{\beta}(v)}\left((1-\overline{\mu}) - \frac{1-\overline{\beta}(v)+\overline{\mu}\overline{\beta}(v)}{1-\overline{\beta}(v)+\mu_+(\hat{s}_A)\overline{\beta}(v)}(1-\mu_+(\hat{s}_A))\right)\overline{\beta}'(v)\\
=&U_v(v,e) - U(v,e)\frac{\mu_+(\hat{s}_A)-\overline{\mu}}{(1-\overline{\beta}(v)+\overline{\mu}\overline{\beta}(v))(1-\overline{\beta}(v)+\mu_+(\hat{s}_A)\overline{\beta}(v))}\overline{\beta}'(v)\\
\geq&U_v(v,e) - U(v,e)\frac{(\mu_+(\hat{s}_A)-\overline{\mu})}{\mu_+(\hat{s}_A)\overline{\mu}}\overline{\beta}'(v)\\
=&U_v(v,e) + U(v,e)\frac{(\mu_+(\hat{s}_A)-\overline{\mu})}{\mu_+(\hat{s}_A)\overline{\mu}}\Big(y(\xi(v)|v)\xi'(v) + Y_v(\xi(v)|v)\Big)\\
\geq&U_v(v,e) - U(v,e)\frac{(\mu_+(\hat{s}_A)-\overline{\mu})}{\mu_+(\hat{s}_A)\overline{\mu}}\left(y(\xi(v)|v)\frac{V_v(v,\xi(v))}{V_{e'}(v,\xi(v))} - Y_v(\xi(v)|v)\right)\\
\geq&0,
\end{aligned}$$

where the second equality is obtained by substituting $H_A(v,e,\overline{\beta}(v)) = 0$, the first inequality follows since $\overline{\mu}, \mu_+(\hat{s}_S) \leq 1$, the penultimate equality follows from the fact that $\overline{\beta}(v) = 1 - Y(\xi(v)|v)$, the second inequality follows since the argument in the proof of Theorem 5 implies that $-\xi'(v) \leq \frac{V_v(v,\xi(v))}{V_{e'}(v,\xi(v))}$, and the last inequality follows from the first part of the above condition.

A.8 Proofs of Lemma 3 and Theorems 7 and 8

A.8.1 Proof of Lemma 3

Fix any σ. To prove the optimality of the cutoff strategy, we show that $T'(y|\sigma) > 0$ for any y. Note that

$$\begin{aligned}
T'(y|\sigma) &= P_A(y|\sigma) + P_B(y|\sigma) + yP_A'(y|\sigma) - (1-y)P_B'(y|\sigma)\\
&\geq y\big[P_A(y|\sigma) + P_A'(y|\sigma)\big] + (1-y)\big[P_B(y|\sigma) - P_B'(y|\sigma)\big]\\
&= y\int_0^1 q_A(s)\big[l(s|y) + l_y(s|y)\big]ds + (1-y)\int_0^1 q_B(s)\big[l(s|y) - l_y(s|y)\big]ds.
\end{aligned}$$

Observe that

$$l(s|y) + l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left[1 + \frac{k_y(y|s)}{k(y|s)} - \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} > -\delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s)ds}{\int_0^1 k(y|s)ds} < \delta$$

because $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$. Similarly,

$$l(s|y) - l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left[1 - \frac{k_y(y|s)}{k(y|s)} + \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right] > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} < \delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)} k(y|s)ds}{\int_0^1 k(y|s)ds} > -\delta.$$

Therefore, we have that $T'(y|\sigma) > 0$ since $\delta \leq \frac{1}{2}$.

It remains to show that there exists an equilibrium in cutoff strategy. Let $\hat{y}$ be a cutoff. Then, we have $n_A(s|\hat{y}) = \int_{\hat{y}}^1 k(y|s) = 1 - K(\hat{y}|s)$. Hence,

$$P_A(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{1-K(\hat{y}|s)}, 1\right\} l(s|y)ds \quad \text{and} \quad P_B(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|y)ds,$$

Now, let

$$T(y|\hat{y}) := yP_A(y|\hat{y}) - (1-y)P_B(y|\hat{y}).$$

Note that

$$T(0|\hat{y}) = -P_B(0|\hat{y}) = -\int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|0)ds < 0,$$

where the inequality holds since $\min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} > 0$ and $l(s|0) \geq 0$ for all s, and $l(s|0) > 0$ for a positive measure of states. Similarly, $T(1|\hat{y}) > 0$. By the continuity of $T(\cdot|\hat{y})$, there is a $\tilde{y}$ such that $T(\tilde{y}|\hat{y}) = 0$. Moreover, such a $\tilde{y}$ is unique since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$.

Next, let $\tau : [0,1] \to [0,1]$ be the map from $\hat{y}$ to $\tilde{y}$, which is implicitly defined by $T(\tau(\hat{y})|\hat{y}) = 0$ according to the Implicit Function Theorem (since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$). Since $P_A(y|\cdot)$ is nondecreasing and $P_B(y|\cdot)$ is nonincreasing $\hat{y}$, $\tau(\cdot)$ is decreasing. Therefore, there is a fixed point such that $\tau(\hat{y}) = \hat{y}$, and hence there is $\hat{y}$ such that $T(\hat{y}|\hat{y}) = 0$.

A.8.2 Proof of Theorem 7

We first show $\hat{y} < 1$. Suppose $\hat{y} = 1$. Then, $n_A(s|1) = 1 - K(1|s) = 0$, so $P_A(y|\hat{y}) = 1$ for any y. Hence, $T(1|1) = P_A(1|1) = 1$, which contradicts the fact that $T(\hat{y}|\hat{y}) = 0$.

We now show that $\hat{y} > \frac{1}{2}$ whenever $\mu(s) > \frac{1}{2}$. Suppose to the contrary $\hat{y} \leq \frac{1}{2}$. We then have $\frac{1}{2} < \mu(s) = 1 - K(\frac{1}{2}|s) \leq 1 - K(\hat{y}|s)$, so $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. Therefore,

$$P_A(y|\hat{y}) - P_B(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{1 - K(\hat{y}|s)}, 1\right\} l(s|y)ds - \int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|y)ds \leq 0. \quad \text{(A.8.1)}$$

Hence, if $\hat{y} < \frac{1}{2}$, then

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1 - \hat{y})P_B(\hat{y}|\hat{y}) < \frac{1}{2}\left[P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\right] \leq 0, \quad \text{(A.8.2)}$$

where the first inequality holds since $\hat{y} < \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) < 0$, a contradiction. Suppose now $\hat{y} = \frac{1}{2}$. Notice that since $K(\hat{y}|s) < 1 - K(\hat{y}|s)$, we have $K(\frac{1}{2}|s) < \frac{1}{2} < 1 - \kappa$, where the the second inequality holds since $\kappa < \frac{1}{2}$. So, $\kappa/(1 - K(\frac{1}{2}|s)) < 1$. Therefore, the last inequality of (A.8.1) becomes strict, and hence

$$T(\hat{y}|\hat{y}) = \frac{1}{2}\left[P_A(\tfrac{1}{2}|\tfrac{1}{2}) - P_B(\tfrac{1}{2}|\tfrac{1}{2})\right] < 0,$$

a contradiction again.

Lastly, let $\mu(s) = \frac{1}{2}$. If $\hat{y} < \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) < 1 - K(\frac{1}{2}|s)$, so we have $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. By (A.8.1) and (A.8.2), we reach a contradiction. If $\hat{y} > \frac{1}{2}$, then $\frac{1}{2} = \mu(s) = 1 - K(\frac{1}{2}|s) > 1 - K(\hat{y}|s)$ and so $K(\hat{y}|s) > 1 - K(\hat{y}|s)$. We then have $P_A(y|\hat{y}) - P_B(y|\hat{y}) \geq 0$ and

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1 - \hat{y})P_B(\hat{y}|\hat{y}) > \frac{1}{2}\left[P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\right] \geq 0,$$

where the first inequality holds since $\hat{y} > \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) > 0$, a contradiction again.

A.8.3 Proof of Theorem 8

For the first part of the theorem, observe that for a given s, justified envy arises whenever $c_A(s) \neq c_B(s)$ as depicted in Figure 5.1. We thus show that there is a positive measure of states in which $c_A(s) \neq c_B(s)$. Suppose to the contrary $c_A(s) = c_B(s)$ for almost all s. Recall that equilibrium admission cutoff of each college satisfies

$$G(c_A(s)) = \max\left\{1 - \frac{\kappa}{1 - K(\hat{y}|s)}, 0\right\} \quad \text{and} \quad G(c_B(s)) = \max\left\{1 - \frac{\kappa}{K(\hat{y}|s)}, 0\right\}.$$

Since $G(\cdot)$ is strictly increasing, if $c_A(s) = c_B(s)$, then we must have either $n_i(s) < \kappa$ for all $i = A, B$ (so that $c_A(s) = c_B(s) = 0$) or $n_A(s) = n_B(s) \geq \kappa$.

First, we cannot have $n_i(s) < \kappa$ for all i in equilibrium, since this means that all applicants are admitted by either college, and this contradicts to $2\kappa < 1$. Second, suppose $n_A(s) = n_B(s) \geq \kappa$. This implies that $K(\hat{y}|s) = \frac{1}{2}$ for all s (recall that $n_A(s) = 1 - K(\hat{y}|s)$ and $n_B(s) = K(\hat{y}|s)$). However, by (5.1), we have $K(\hat{y}|s') < K(\hat{y}|s)$ for all $s' > s$. Therefore, we reach a contradiction again.

To see the second part of the theorem, recall that for given $\hat{y}$ in equilibrium, the mass of students applying to B is $K(\hat{y}|s)$. Thus, if there is a positive measure of states in which $K(\hat{y}|s) < \kappa$, college B faces under-subscription in such states. Therefore, the equilibrium outcome is inefficient.

A.9 Proof of Theorem 9

Suppose there is a symmetric equilibrium as described in the theorem. Then, colleges A and B will admit all acceptable students with $v > \tilde{v}$, where $\tilde{v}$ is such that each of A and B fills its capacity in the popular state, i.e., $s_a(1-\varepsilon)[1-G(\tilde{v})] = \kappa$ and $(1-s_b)(1-\varepsilon)[1-G(\tilde{v})] = \kappa$, and wait-lists the remaining students. College C will offer admissions to all of these students (i.e., those whose scores are above $\tilde{v}$), knowing that exactly measure ε^2 of them will accept its offer. It will also offer $\kappa - \varepsilon^2$ admissions to all students with $v \in [\hat{v}, \tilde{v}]$, where $\hat{v}$ is such that $G(\tilde{v}) - G(\hat{v}) = \kappa - \varepsilon^2$.

The students in $[\hat{v}, \tilde{v}]$ now have a choice to make. If a student accepts C, then she will get u'' for sure, but if she turns down C's offer, then with probability $1-\varepsilon$ the less popular one between A and B will offer an admission to her (assuming all other students admitted by C have accepted that offer), and the student will earn the payoff u if she happens to like the college, or u' otherwise. Since $u'' > (1-\varepsilon)u$, she will accept C.

Given this, consider now the incentive for deviation of college A. If it does not deviate, there will be seats left in the less popular state, equal to $\kappa - s_b(1-\varepsilon)[1-G(\hat{v})]$. Thus, A will fill those vacant seats with students whose scores are below $\hat{v}$. Thus, its payoff is

$$
\begin{aligned}
\pi_A &= \frac{1}{2}s_a(1-\varepsilon)\int_{\tilde{v}}^{1} v\,dG(v) + \frac{1}{2}\Big[s_b(1-\varepsilon)\int_{\tilde{v}}^{1} v\,dG(v) + (1-\varepsilon)\int_{\check{v}}^{\hat{v}} v\,dG(v)\Big] \\
&= \frac{1}{2}(1-\varepsilon)\Big[\int_{\tilde{v}}^{1} v\,dG(v) + \int_{\check{v}}^{\hat{v}} v\,dG(v)\Big],
\end{aligned}
$$

where $\check{v}$ is such that

$$
(1-\varepsilon)[G(\hat{v}) - G(\check{v})] = \kappa - s_b(1-\varepsilon)[1-G(\tilde{v})]. \tag{A.9.1}
$$

and the second equality follows from $s_a = 1 - s_b$.

Suppose now A admits a small fraction, say δ', of (acceptable) students just below $\tilde{v}$ instead of admitting those who are acceptable and slightly above $\tilde{v}$, say $[\tilde{v}, \tilde{v}+\delta]$, where δ and δ' are such that

$$
G(\tilde{v}+\delta) - G(\tilde{v}) = G(\tilde{v}) - G(\tilde{v}-\delta'). \tag{A.9.2}
$$

Notice that students in $[\tilde{v}-\delta', \tilde{v}]$ accept A's admission offer, since they prefer it over C. Hence, A's payoff under the deviation is

$$\begin{aligned}
\pi_A^d &= (1-\varepsilon)\int_{\tilde{v}-\delta'}^{\tilde{v}} v\, dG(v) + \frac{1}{2}s_a(1-\varepsilon)\int_{\tilde{v}+\delta}^{1} v\, dG(v) \\
&\quad + \frac{1}{2}\Big[s_b(1-\varepsilon)\int_{\tilde{v}+\delta}^{1} v\, dG(v) + (1-\varepsilon)\int_{\overline{v}}^{\hat{v}} v\, dG(v)\Big] \\
&= (1-\varepsilon)\int_{\tilde{v}-\delta'}^{\hat{v}} v\, dG(v) + \frac{1}{2}(1-\varepsilon)\Big[\int_{\tilde{v}+\delta}^{1} v\, dG(v) + \int_{\tilde{v}}^{\hat{v}} v\, dG(v)\Big],
\end{aligned}$$

where $\overline{v}$ satisfies

$$(1-\varepsilon)[G(\hat{v}) - G(\overline{v})] = \kappa - (1-\varepsilon)[G(\tilde{v}) - G(\tilde{v}-\delta')] - s_b(1-\varepsilon)[1 - G(\tilde{v}+\delta)],$$

that is, $\overline{v}$ is set to meet the capacity in the less popular state. Observe that $\overline{v} > \check{v}$, since

$$\begin{aligned}
(1-\varepsilon)[G(\hat{v}) - G(\overline{v})] &= \kappa - s_b(1-\varepsilon)[1 - G(\hat{v})] - s_a(1-\varepsilon)[G(\tilde{v}) - G(\tilde{v}-\delta')] \\
&= (1-\varepsilon)[G(\hat{v}) - G(\check{v})] - s_a(1-\varepsilon)[G(\tilde{v}) - G(\tilde{v}-\delta')], \qquad \text{(A.9.3)}
\end{aligned}$$

where the first equality follows from (A.9.2) and the fact that $s_a = 1 - s_b$, and the last equality follows from (A.9.1). Thus, we have

$$\begin{aligned}
\frac{2(\pi_A^d - \pi_A)}{1-\varepsilon} &= 2\int_{\tilde{v}-\delta'}^{\tilde{v}} v\, dG(v) - \Big[\int_{\tilde{v}}^{\tilde{v}+\delta} v\, dG(v) + \int_{\check{v}}^{\overline{v}} v\, dG(v)\Big] \\
&= 2\Big[\tilde{v}\, G(\tilde{v}) - (\tilde{v}-\delta)G(\tilde{v}-\delta') - \int_{\tilde{v}-\delta'}^{\tilde{v}} G(v)dv\Big] \\
&\quad - \Big[(\tilde{v}+\delta)G(\tilde{v}+\delta) - \tilde{v}\, G(\tilde{v}) - \int_{\tilde{v}}^{\tilde{v}+\delta} G(v)dv\Big] - \Big[\overline{v}\, G(\overline{v}) - \check{v}\, G(\check{v}) - \int_{\check{v}}^{\overline{v}} G(v)dv\Big] \\
&= \tilde{v}\Big[G(\tilde{v}) - G(\tilde{v}-\delta')\Big] + 2\Big[\delta' G(\tilde{v}-\delta') - \int_{\tilde{v}-\delta'}^{\tilde{v}} G(v)dv\Big] \\
&\quad - \Big[\delta G(\tilde{v}+\delta) - \int_{\tilde{v}}^{\tilde{v}+\delta} G(v)dv\Big] - \Big[\overline{v}\, G(\overline{v}) - \check{v}\, G(\check{v}) - \int_{\check{v}}^{\overline{v}} G(v)dv\Big] \\
&\geq (\tilde{v} - 2\delta')\Big[G(\tilde{v}) - G(\tilde{v}-\delta')\Big] - \delta\Big[G(\tilde{v}+\delta) - G(\tilde{v})\Big] - \overline{v}\Big[G(\overline{v}) - G(\check{v})\Big]
\end{aligned}$$

where the second equality follows from the integration by parts, and the third equality follows from (A.9.2). The inequality holds since $\int_{\tilde{v}-\delta'}^{\tilde{v}} G(v) \leq \delta' G(\tilde{v})$, $\int_{\tilde{v}}^{\tilde{v}+\delta} G(v)dv \geq \delta G(\tilde{v})$ and $\int_{\check{v}}^{\overline{v}} G(v) \geq (\overline{v} - \check{v})G(\check{v})$. Observe that by rearranging (A.9.3), we have $G(\overline{v}) - G(\check{v}) = s_a\Big[G(\tilde{v}) - G(\tilde{v}-\delta')\Big]$. Hence, using (A.9.2) again, we get

$$\frac{2(\pi_A^d - \pi_A)}{1-\varepsilon} \geq \Big[G(\tilde{v}) - G(\tilde{v}-\delta')\Big](\tilde{v} - 2\delta' - \delta - \overline{v}\, s_a) = \Big[G(\tilde{v}) - G(\tilde{v}-\delta')\Big]\Big(s_a(\tilde{v} - \overline{v}) + s_b\, \tilde{v} - (2\delta' + \delta)\Big).$$

Therefore, for sufficiently small δ, we have $\pi_A^d > \pi_A$.

A.10 Proof of Lemma 4

Recall that when both colleges A and B report truthfully up to the capacity, they achieve jointly optimal matching for the two colleges. Now suppose college A unilaterally deviates by either reporting untruthfully about its preferences or its capacity and is strictly better off for some state s. Then, college B must be strictly worse off. Thus, there must exist a positive measure set of students whom A must obtain from the deviation which it prefers to some students it had before the deviation. At the same time, it must be the case that either college B gets a positive measure set of students who are worse than the former set of students or it has some unfilled seats left after A's deviation. Note that students in the former set (who are assigned to A in the new matching) must prefer B, or else the original matching would be not be stable. But then since B prefer each of those students to some students it has in the new matching, this means that the new matching is not stable (given the stated preferences).

B Appendix B: More than Two Colleges

Our main model in Section 2 considers the case with two colleges. In this section, we show that our analysis extends to the case with more than two colleges. While the extension works for any arbitrary number of colleges, we provide the result for the three-college case for expositional simplicity. It will become clear that the method also extends to larger numbers.

Let $\sigma_i : \mathcal{V} \to [0,1]$ be college i's admission strategy, where $i = 1,2,3$. In each state $s \in [0,1]$, let $\mu_{ijk}(s)$, where $i,j,k = 1,2,3$, denote the mass of students whose preference ordering is $i \succ j \succ k$. Define the following notations.

- $\mu_{i\succ j}(s) := \mu_{ijk}(s) + \mu_{ikj}(s) + \mu_{kij}(s)$ (the mass of students who prefer i over j in state s),
- $\mu_{i\succ j,k}(s) := \mu_{ijk}(s) + \mu_{ikj}(s)$ (the mass of students who prefer i over j and k in state s),

and

$$\overline{\mu}_{i\succ j} := \int_0^1 \mu_{i\succ j}(s)\,ds, \quad \overline{\mu}_{i\succ j,k} := \int_0^1 \mu_{i\succ j,k}(s)\,ds.$$

For given $\sigma_i(\cdot)$, $i = 1,2,3$, let $n_i(v)$ be the probability that a student with score v attends college i in state s when she is admitted by i. That is,

$$n_i(v|s) := \prod_{t=j,k}(1-\sigma_t(v)) + \mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v)) + \mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v)) + \mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v). \tag{B.0.1}$$

The student will attend college i if she is admitted only by i, which happens with probability $(1-\sigma_j(v))(1-\sigma_k(v))$; or is admitted by college i and one of the less preferred colleges, which happens with probability $\mu_{i\succ j}(s)\sigma_j(v)(1-\sigma_k(v))+\mu_{i\succ k}(s)\sigma_k(v)(1-\sigma_j(v))$ in state s; or is admitted by both of the other colleges but prefers i the most, which happens with probability $\mu_{i\succ j,k}(s)\sigma_j(v)\sigma_k(v)$ in state s.

Thus, for a given profile of admission strategies, $\sigma = (\sigma_i)_{i=1,2,3}$, in equilibrium, the mass of students who attend college i in state s is

$$m_i(s) := \int_0^1 \sigma_i(v)\, n_i(v|s)\, dG(v),$$

and college i's payoff is

$$\pi_i = \int_0^1 v\, \sigma_i(v)\, \overline{n}_i(v)\, dG(v) - \lambda \int_0^1 \max\{m_i(s) - \kappa, 0\}\, ds, \tag{B.0.2}$$

where

$$\overline{n}_i(v) := \int_0^1 n_i(v|s)\, ds. \tag{B.0.3}$$

Recall that in the two-school case, the monotonicity of $\mu(\cdot)$ yields cutoff states $(\hat{s}_A, \hat{s}_B)$ that trigger over-enrollment for each college, and the set of over-demanded states for each of them is a connected interval, $(\hat{s}_A, 1]$ and $[0, \hat{s}_B)$. Using this, we project the admission strategies to state space in order to establish the existence of MME. This allows us to use the Brouwer's fixed point theorem. When there are more than two colleges, however, we do not know the structure of the set of over-demanded states in general, so we cannot directly define a map from cutoff states to cutoff states. Nonetheless, the main idea of the proof can be carried over, although we use a fixed point theorem (Schauder) in a functional space.

Define a **subdistribution** $F_i : [0,1] \to [0,1]$, $i = 1,2,3$, such that $F_i(0) = 0$ and

$$F_i(s) := \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s\big). \tag{B.0.4}$$

The subdistribution of college i places a positive mass only on the states in which college i is over-demanded. Observe that $F_i(\cdot)$ is nondecreasing and

$$0 \le F_i(s') - F_i(s) \le s' - s, \quad \forall\, s' \ge s.$$[25]

[25]The second inequality holds because

$$\begin{aligned} F_i(s') - F_i(s) &= \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s'\big) - \text{Prob}\big(m_i(t) > \kappa \text{ for } t < s\big) \\ &= \text{Prob}\big(m_i(t) > \kappa \text{ for } s < t < s'\big) \\ &\le \text{Prob}(s < t < s') \\ &= s' - s. \end{aligned}$$

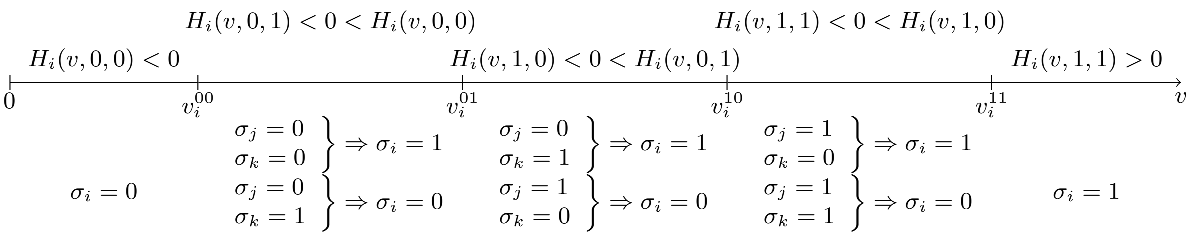


Figure B.1: College i's Admission Decision

Let $\mathcal{F}_i$ be the set of all such subdistributions and $\mathcal{F} := \times_{i=1}^{3} \mathcal{F}_i$. (It will become clear that these subdistributions will play a similar role to the cutoff states in the two-school case.)

Using the subdistributions, each college's payoff is now given by[26]

$$\pi_i = \int_0^1 v\,\sigma_i(v)\,\overline{n}_i(v)\,dG(v) - \lambda \int_0^1 \big(m_i(s) - \kappa\big)\,dF_i(s) \qquad \text{(B.0.5)}$$
$$= \int_0^1 \sigma_i(v)\,H_i(v, \sigma_j(v), \sigma_k(v))\,dG(v) + \lambda \int_0^1 \kappa\,dF_i(s),$$

where

$$H_i(v, \sigma_j(v), \sigma_k(v)) := v\,\overline{n}_i(v) - \lambda \int_0^1 n_i(v|s)\,dF_i(s) \qquad \text{(B.0.6)}$$

is college i's marginal payoff from admitting a student with score v. Note that this marginal payoff depends on the subdistribution F_i, as $\overline{n}_i(v)$ is a constant for given admission strategies $(\sigma_i)_{i=1,2,3}$ (by (B.0.3)) and $n_i(v)$ is evaluated by the subdistribution.

Note that (B.0.6) can be decomposed as follow:

$$H_i(v, \sigma_j(v), \sigma_j(v)) = (1-\sigma_j(v))(1-\sigma_k(v))H_i(v,0,0) + \sigma_j(v)(1-\sigma_k(v))H_i(v,1,0)$$
$$+ (1-\sigma_j(v))\sigma_k(v)H_i(v,0,1) + \sigma_j(v)\sigma_k(v)H_i(v,1,1),$$

where $H_i(v,0,0)$ is college i's marginal payoff from admitting a student with score v if she is refused by both of the other colleges, $H_i(v,1,0)$ and $H_i(v,0,1)$ are the marginal payoffs if the student is admitted by college j (k) but rejected by k (j, respectively), and $H_i(v,1,1)$ is the marginal payoff if the student is admitted by both of the other colleges.

Let us now define v_i^{11}, v_i^{10}, v_i^{01} and v_i^{00} such that

$$H_i(v_i^{11},1,1) = 0, \quad H_i(v_i^{10},1,0) = 0, \quad H_i(v_i^{01},0,1) = 0, \quad H_i(v_i^{00},0,0) = 0.$$

Similar to the two-school case, $H_i(v, \sigma_j, \sigma_k)$ partitions the students' type space. College i admits type v students for sure if $H_i(v,1,1) > 0$ and rejects them if $H_i(v,0,0) < 0$. In the

[26] Note that since F_i is Lipschitz continuous, so it is absolute continuous. Thus, the integration is well defined. Observe also that (B.0.5) does not involve $\max\{\cdot,\cdot\}$ in the cost (see (B.0.2) for comparison), as the subdistribution is defined for states where $m_i(s) > \kappa$ by (B.0.4), and the college's cost is evaluated by the subdistribution.

case $H_i(v,1,1) < 0 < H_i(v,0,0)$, college i admits type v students only when $H_i(v,1,0) > 0$ or $H_i(v,0,1) > 0$; that is, those students are worthy only in the case that they is admitted by one of the other colleges. This shows that colleges engage in *strategic targeting* for those intermediate range of scores.

Randomization may emerge for some students. For students with v such that

$$\max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\},$$

all three colleges engage in mixed-strategies, where the mixed-strategies satisfy

$$H_i(v, \sigma_j(v), \sigma_k(v)) = 0 \quad \forall i,j,k = 1,2,3.$$

For students with v such that $H_k(v,0,0) < 0$ and

$$\max\{H_i(v,1,0), H_j(v,1,0)\} < 0 < \min\{H_i(v,0,0), H_j(v,0,0)\},$$

college k does not admit such students, but colleges i and j engage in mixed-strategies satisfying

$$H_i(v,\sigma_j,0) = 0 \quad \text{and} \quad H_j(v,\sigma_i,0) = 0.$$

A typical mixed-strategy equilibrium is depicted in Figure B.2 when, for instance,

$$v_3^{00} < v_2^{00} < v_1^{00} < v_3^{01} < v_2^{01} < v_1^{01} < v_3^{10} < v_2^{10} < v_1^{10} < v_3^{11} < v_2^{11} < v_1^{11}.$$

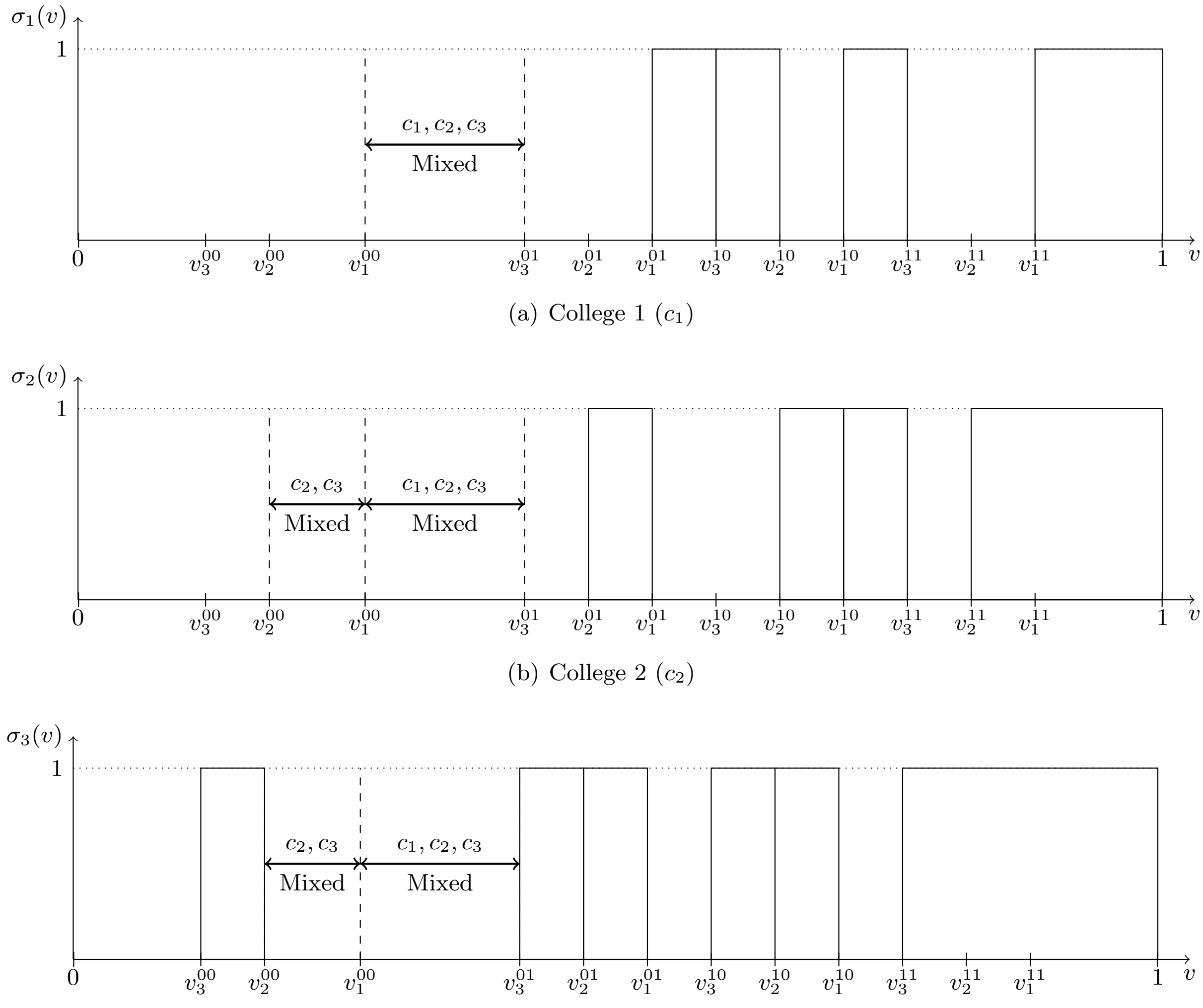
Note that, as in the two-school case, there are many ways that colleges could coordinate (even in a mixed-strategy equilibrium). Hence, we consider the maximally mixed-strategy as before and provide the existence of such equilibrium.

For a given profile of subdistributions $(F_i)_{i=1}^3$, let $\sigma := (\sigma_i)_{i=1}^3$ be the profile of admission strategies that satisfy the local conditions described above. Then, such σ in turn determines a new profile of subdistributions, $(F_i)_{i=1}^3$ via (B.0.4). Next, we define $T : \mathcal{F} \to \mathcal{F}$, a self-map from the set of subdistributions to itself, where $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$. The existence of equilibrium is achieved when T has a fixed point (on the functional space of $\mathcal{F}$).

As mentioned earlier, the idea of proving the existence of equilibrium is similar to the idea of Theorem 3, projecting the strategy profile into a simpler space. The difference is that in the two-school case, the strategy profiles are projected into the state space, but in the general case, they are projected into the set of subdistributions $\mathcal{F}$.

Theorem 11. *There exists an equilibrium with maximally mixed-strategies.*

We first show that $\mathcal{F}$ is a compact and convex subset of a normed linear space, and $T : \mathcal{F} \to \mathcal{F}$



(a) College 1 (c_1)

(b) College 2 (c_2)

(c) College 3 (c_3)

Figure B.2: Admission Strategies

is continuous. Then, T has a fixed point by Schauder's fixed point theorem.[27] We then show that the identified strategies indeed constitute mutual (global) best responses. We provide a formal proof in the next subsection.

B.1 Proof of Theorem 11

For given $(F_i)_{i=1,2,3}$, consider colleges' strategy profile $(\sigma_i)_{i=1,2,3}$ which satisfies the following local conditions:

- $\sigma_i(v) = 1$ if $H_1(v,1,1) > 0$, $i = 1,2,3$.

- $\sigma_1(v) = 0$ if $H_1(v,1,1) < 0$, $H_2(v,1,1) > 0$, $H_3(v,1,1) > 0$.
  $\sigma_2(v) = 0$ if $H_1(v,1,1) > 0$, $H_2(v,1,1) < 0$, $H_3(v,1,1) > 0$.
  $\sigma_3(v) = 0$ if $H_1(v,1,1) > 0$, $H_2(v,1,1) > 0$, $H_3(v,1,1) < 0$.

- $\sigma_1(v) = 0,\ \sigma_2(v) = 1,\ \sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,1) > 0 \\ H_3(v,1,1) < 0, H_3(v,0,1) > 0 \end{cases}$

- $\sigma_1(v) = 1,\ \sigma_2(v) = 0,\ \sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,1) > 0 \\ H_2(v,1,1) < 0 \\ H_3(v,1,1) < 0, H_3(v,1,0) > 0 \end{cases}$

- $\sigma_1(v) = 1,\ \sigma_2(v) = 1,\ \sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,1,0) > 0 \\ H_2(v,1,1) < 0, H_2(v,1,0) > 0 \\ H_3(v,1,1) < 0 \end{cases}$

- $\sigma_1(v) = 1,\ \sigma_2(v) = 0,\ \sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,0) > 0 \\ H_2(v,1,1) < 0, H_2(v,1,0) < 0 \\ H_3(v,1,1) < 0, H_3(v,1,0) < 0 \end{cases}$

- $\sigma_1(v) = 0,\ \sigma_2(v) = 1,\ \sigma_3(v) = 0$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,1,0) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,0) > 0 \\ H_3(v,1,1) < 0, H_3(v,0,1) < 0 \end{cases}$

[27] Schauder's fixed point theorem is a generalization of Brouwer's theorem on a normed linear space. It guarantees that every continuous self-map on a nonempty, compact, convex subset of a normed linear space has a fixed point (see Ok, 2007).

- $\sigma_1(v) = 0$, $\sigma_2(v) = 0$, $\sigma_3(v) = 1$ if $\begin{cases} H_1(v,1,1) < 0, H_1(v,0,1) < 0 \\ H_2(v,1,1) < 0, H_2(v,0,1) < 0 \\ H_3(v,1,1) < 0, H_3(v,0,0) > 0 \end{cases}$

- $\sigma_i(v) = 0$ if $H_1(v,0,0) < 0$, $i = 1,2,3$.

- $\sigma_i(v)$'s satisfy $H_1(v,\sigma_2(v),\sigma_3(v)) = H_2(v,\sigma_1(v),\sigma_3(v)) = H_3(v,\sigma_1(v),\sigma_2(v)) = 0$, if

$$\max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\}$$

- $\sigma_i(v)$ and $\sigma_j(v)$ satisfy $H_i(v,\sigma_j,0) = 0$ and $H_j(v,\sigma_i,0) = 0$ if $H_k(v,0,0) < 0$ and

$$\max \{H_i(v,1,0), H_j(v,1,0)\} < 0 < \min \{H_i(v,0,0), H_j(v,0,0)\}$$

Now, let $\mathbf{CB}([0,1])$ be the space of continuous and bounded real maps on $[0,1]$. Then, $\mathbf{CB}([0,1])$ is a normed linear space, with a sup norm $\|\cdot\|$, i.e., for any $F, F' \in \mathbf{CB}([0,1])$,

$$\left\|F - F'\right\| = \sup_{s\in[0,1]} \left|F(s) - F'(s)\right|.$$

Lemma B1. *$\mathcal{F}$ is compact and convex.*

Proof. We first show that $\mathcal{F}_i$, $i = 1,2,3$, is closed. To this end, consider any sequence $\{F_i^n\}$, where $F_i^n \in \mathcal{F}_i$ for each n, such that $\|F_i^n - F_i\| \to 0$ as $n \to \infty$. We prove that $F_i \in \mathcal{F}_i$.

Observe first that F_i is nondecreasing. Suppose to the contrary that $F_i(s') - F_i(s) < 0$ for some $s' > s$. But then,

$$\begin{aligned} \|F_i^n - F_i\| &\geq \max \left\{\left|F_i^n(s') - F_i(s')\right|, \left|F_i(s) - F_i^n(s)\right|\right\} \\ &\geq \tfrac{1}{2}(\left|F_i^n(s') - F_i(s')\right| + F_i(s) - F_i^n(s)) \\ &\geq \tfrac{1}{2}\left|F_i^n(s') - F_i(s') + F_i(s) - F_i^n(s)\right| \\ &\geq \tfrac{1}{2}\left|F_i(s) - F_i(s')\right| \\ &> 0 \end{aligned}$$

which is a contradiction. Likely, for $s' > s$, we must have that $F_i(s') - F_i(s) \leq s' - s$. If $F_i(s') - F_i(s) > s' - s$, then

$$\begin{aligned} \|F_i^n - F_i\| &\geq \max \left\{\left|F_i(s') - F_i^n(s')\right|, \left|F_i^n(s) - F_i(s)\right|\right\} \\ &\geq \tfrac{1}{2}(\left|F_i(s') - F_i^n(s')\right| + \left|F_i^n(s) - F_i(s)\right|) \end{aligned}$$

$$
\begin{aligned}
&\geq \tfrac{1}{2}\left|F_i(s') - F_i(s) + F_i^n(s) - F_i^n(s')\right| \\
&\geq \tfrac{1}{2}\left|F_i(s') - F_i(s) - (s'-s)\right| \\
&> 0,
\end{aligned}
$$

which is a contradiction again. Combining these, we have $F_i \in \mathcal{F}_i$, proving that $\mathcal{F}_i$ is closed.

Next, we show that $\mathcal{F}_i$ is compact. Note that for any $F_i \in \mathcal{F}_i$ and $s, s' \in [0,1]$,

$$
\left|F_i(s') - F_i(s)\right| \leq \left|s' - s\right|,
$$

Hence, $\mathcal{F}_i$ is Lipschitz continuous and so is equicontinuous and bounded. By the Arzèla-Ascoli theorem,[28] $\mathcal{F}_i$ is compact.

We now show that $\mathcal{F}_i$ is convex. Observe that for any $F_i, F_i' \in \mathcal{F}$ and $s, s' \in [0,1]$, for and $\eta \in (0,1)$,

$$
\begin{aligned}
(\eta F_i + (1-\eta)F_i')(s') - (\eta F_i + (1-\eta)F_i')(s) &= \eta(F_i(s') - F_i(s)) + (1-\eta)(F_i'(s') - F_i'(s)) \\
&\leq \eta(s'-s) + (1-\eta)(s'-s) \\
&= s' - s,
\end{aligned}
$$

which proves that $\mathcal{F}_i$ is convex.

Since $\mathcal{F}_i$ is compact and closed, so is its Cartesian product $\mathcal{F} = \times_{i=1}^3 \mathcal{F}_i$ (with respect to the product topology). ■

Lemma B2. *T is continuous.*

Proof. The proof involves several steps:

Step 1. *v_i^{jk}'s are continuous on F_1, F_2, F_3.*

Proof . We first show that v_i^{jk}'s are continuous in F_i. Fix any $F_i \in \mathcal{F}_i$ and $\varepsilon > 0$. Take $\delta = \frac{\overline{\mu}_{i\succ j,k}}{2\lambda}\varepsilon$. Then, for any $F_i, F_i' \in \mathcal{F}_i$ such that $\|F_i - F_i'\| < \delta$, we have that

$$
\begin{aligned}
\left|v_i^{jk} - v_i^{jk\prime}\right| &= \left|\frac{\lambda}{\overline{\mu}_{i\succ j,k}} \int_0^1 \mu(s)_{i\succ j,k}[dF_i(s) - dF_i'(s)]\right| \\
&= \frac{\lambda}{\overline{\mu}_{i\succ j,k}} \left|\mu_{i\succ j,k}(1)[F_i(1) - F_i'(1)] - \int_0^1 \mu'_{i\succ j,k}(s)[F_i(s) - F_i'(s)]ds\right| \\
&\leq 2\left\|F_i(s) - F_i'(s)\right\| \\
&< \varepsilon,
\end{aligned}
$$

[28] Arzèla-Ascoli theorem gives conditions for a set of $\mathbf{C}(T)$ to be compact, where $\mathbf{C}(T)$ is the space of continuous maps on T and T is a compact metric space. A subset of $\mathbf{C}(T)$ is compact if and only if it is closed, bounded, and equicontinuous.

where the third equality follows from the integration by parts and $F_i(0) = F_i'(0) = 0$, and the first inequality holds since $\int_0^1 \mu'_{i \succ j,k}(s)ds = \mu_{i \succ j,k}(1) - \mu_{i \succ j,k}(0) \leq 1$. $\square$

Step 2. *σ_i's in mixed-strategies are continuous.*

Proof . Consider, at first, students with score v such that

$$H_k(v, 0, 0) < 0, \tag{B.1.1}$$

$$H_i(v, 1, 0) < 0 < H_i(v, 0, 0), \tag{B.1.2}$$

$$H_j(v, 1, 0) < 0 < H_j(v, 0, 0). \tag{B.1.3}$$

That is, college k puts zero probability for those students (by (B.1.1)), and colleges i and j use mixed-strategies σ_i and σ_j which satisfy $H_i(v, \sigma_j, 0) = 0$ and $H_j(v, \sigma_i, 0) = 0$.

Now, let $J_i : [0,1]^2 \times [0,1]^2 \to [0,1]$ such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) \equiv H_i(v, \sigma_j, 0) = v\big[(1-\sigma_j) + \overline{\mu}_{i \succ j}\sigma_j(v)\big] - \lambda \int_0^1 \big[(1-\sigma_j) + \mu_{i \succ j}(s)\sigma_j(v)\big] dF_i(s),$$

$$J_j(F_i, F_j, \sigma_i, \sigma_j) \equiv H_j(v, \sigma_i, 0) = v\big[(1-\sigma_i) + \overline{\mu}_{j \succ i}\sigma_i(v)\big] - \lambda \int_0^1 \big[(1-\sigma_i) + \mu_{j \succ i}(s)\sigma_i(v)\big] dF_j(s).$$

Then, σ_i and σ_j are the solutions to $J_i = 0$ and $J_j = 0$ in terms of F_i and F_j. Observe that

$$J_i = (1-\sigma_j)H_i(v, 0, 0) + \sigma_j H_i(v, 1, 0).$$

Hence,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v, 0, 0) + H_i(v, 1, 0) < 0,$$

where inequality follows from (B.1.2). Similarly, we also have by (B.1.3)

$$\frac{\partial J_j}{\partial \sigma_i} = -H_j(v, 0, 0) + H_j(v, 1, 0) < 0.$$

Therefore,

$$\Delta_{ij} := \begin{vmatrix} \frac{\partial J_i}{\partial \sigma_i} & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & \frac{\partial J_j}{\partial \sigma_j} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_i}{\partial \sigma_j} \\ \frac{\partial J_j}{\partial \sigma_i} & 0 \end{vmatrix} = -\frac{\partial J_i}{\partial \sigma_j}\frac{\partial J_j}{\partial \sigma_i} < 0.$$

Since $\Delta_{ji} \neq 0$, the Implicit function theorem implies that there are unique σ_i and σ_j such that

$$J_i(F_i, F_j, \sigma_i, \sigma_j) = 0 \quad \text{and} \quad J_j(F_i, F_j, \sigma_i, \sigma_j) = 0.$$

Furthermore, such σ_i and σ_j are continuous.

Consider now the case that $H_1(v,\sigma_2,\sigma_3) = H_2(v,\sigma_1,\sigma_3) = H_3(v,\sigma_1,\sigma_2) = 0$ when

$$\max_{i=1,2,3} \{H_i(v,1,0), H_i(v,0,1)\} < 0 < \min_{i=1,2,3} \{H_i(v,0,0)\}. \tag{B.1.4}$$

Similar as before, let

$$\begin{aligned} J_1(F_1,F_2,F_3,\sigma_1,\sigma_2,\sigma_3) &\equiv H_1(v,\sigma_2,\sigma_3) = 0, \\ J_2(F_1,F_2,F_3,\sigma_1,\sigma_2,\sigma_3) &\equiv H_2(v,\sigma_1,\sigma_3) = 0, \\ J_3(F_1,F_2,F_3,\sigma_1,\sigma_2,\sigma_3) &\equiv H_3(v,\sigma_1,\sigma_2) = 0. \end{aligned}$$

Observe that

$$\begin{aligned} J_i &= (1-\sigma_j)(1-\sigma_k)H_i(v,0,0) + \sigma_j(1-\sigma_k)H_i(v,1,0) + (1-\sigma_j)\sigma_k H_i(v,0,1) + \sigma_j\sigma_k H_i(v,1,1) \\ &= (1-\sigma_j)H_i(v,0,0) + \sigma_j(1-\sigma_k)H_i(v,1,0) - (1-\sigma_j)\sigma_k H_k(v,1,0) + \sigma_j\sigma_k H_i(v,1,1). \end{aligned}$$

where the second inequality holds after some rearrangement using the fact that $1-\mu_{i\succ k}(s) = \mu_{k\succ i}(s)$. Therefore,

$$\frac{\partial J_i}{\partial \sigma_j} = -H_i(v,0,0) + (1-\sigma_k)H_i(v,1,0) + \sigma_k H_k(v,1,0) + \sigma_k H_i(v,1,1) < 0,$$

where the inequality holds since $H_i(v,0,0) > 0$, $H_i(v,1,0) < 0$, $H_k(v,1,0) < 0$ and $H_i(v,1,1) < 0$ by (B.1.4). This implies that

$$\Delta := \begin{vmatrix} \frac{\partial J_1}{\partial \sigma_1} & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & \frac{\partial J_2}{\partial \sigma_2} & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & \frac{\partial J_3}{\partial \sigma_3} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial J_1}{\partial \sigma_2} & \frac{\partial J_1}{\partial \sigma_3} \\ \frac{\partial J_2}{\partial \sigma_1} & 0 & \frac{\partial J_2}{\partial \sigma_3} \\ \frac{\partial J_3}{\partial \sigma_1} & \frac{\partial J_3}{\partial \sigma_2} & 0 \end{vmatrix} = \frac{\partial J_1}{\partial \sigma_2}\frac{\partial J_2}{\partial \sigma_3}\frac{\partial J_3}{\partial \sigma_1} + \frac{\partial J_1}{\partial \sigma_3}\frac{\partial J_2}{\partial \sigma_1}\frac{\partial J_3}{\partial \sigma_2} < 0$$

Using the Implicit function theorem again, we conclude that such $\sigma_1, \sigma_2, \sigma_3$ exist and they are continuous. □

Observe that from Step 1 and Step 2, $H_i(v,\sigma_j,\sigma_k)$, $i = 1,2,3$, is continuous in $(F_i)_{i=1,2,3}$ for a given s and fixed v.

Step 3. *$m_i(s)$ is continuous.*

Proof . Consider any $F_i, F_i' \in \mathcal{F}_i$ such that $\|F_i - F_i'\| < \delta$ for all $i = 1,2,3$. Let σ_i and σ_i' are admission strategies of college i which correspond to F_i and F_i', respectively. Then, for a given s and v, $n_i(v|s)$ is defined by (B.0.1) and $n_i'(v|s)$ is defined similarly using σ_i'.

Let $X := \{v \in [0,1] | \, |\sigma_i(v) - \sigma_i'(v)| \geq \varepsilon/2\}$. Clearly,

$$\left|\sigma_i(v) - \sigma_i'(v)\right| = \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_X(v) + \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v),$$

where $\mathbb{1}_X(v)$ is the indicator function which is 1 if $v \in X$ or 0 otherwise, and X^c is the complementary set of X. Since v_i^{jk} are continuous by Step 1, we have

$$\int_0^1 \mathbb{1}_X(v)\, dG(v) < \frac{\varepsilon}{2}. \tag{B.1.5}$$

For $v \in X^c$, it must be the case that either $\sigma_i = \sigma_i'$, or σ_i and σ_i' are the mixed-strategies. Thus, we have for $v \in X^c$,

$$\left|\sigma_i(v) - \sigma_i'(v)\right| < \frac{\varepsilon}{2}. \tag{B.1.6}$$

Observe that

$$\begin{aligned}
\int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \, dG(v) &= \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_X(v)\, dG(v) + \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v)\, dG(v) \\
&< \int_0^1 \mathbb{1}_X(v)\, dG(v) + \int_0^1 \left|\sigma_i(v) - \sigma_i'(v)\right| \mathbb{1}_{X^c}(v)\, dG(v) \\
&< \varepsilon,
\end{aligned}$$

where the first inequality holds since $\sigma_i, \sigma_i' \leq 1$, and the last inequality follows from (B.1.5) and (B.1.6). Thus, there exists δ_1 such that $\|F_i - F_i'\| < \delta_1$, for all $i, i' = 1, 2, 3$, implies

$$\begin{aligned}
&\int_0^1 \left|\sigma_i(1-\sigma_j)(1-\sigma_k) - \sigma_i'(1-\sigma_j')(1-\sigma_k')\right| \, dG(v) \\
&\leq \int_0^1 \left[\left|\sigma_i - \sigma_i'\right| (1-\sigma_j)(1-\sigma_k) + \left|\sigma_j - \sigma_j'\right| \sigma_i'(1-\sigma_k) + \left|\sigma_k - \sigma_k'\right| \sigma_i'(1-\sigma_j')\right] dG(v) \\
&< \frac{\varepsilon}{4}
\end{aligned}$$

Similarly, there are δ_t, $t = 2, 3, 4$, such that $\|F_i - F_i'\| < \delta_t$ respectively imply that

$$\left|\sigma_i\sigma_j(1-\sigma_k) - \sigma_i'\sigma_j'(1-\sigma_k')\right| < \frac{\varepsilon}{4}, \quad \left|\sigma_i\sigma_k(1-\sigma_j) - \sigma_i'\sigma_k'(1-\sigma_j')\right| < \frac{\varepsilon}{4}, \quad \left|\sigma_i\sigma_j\sigma_k - \sigma_i'\sigma_j'\sigma_k'\right| < \frac{\varepsilon}{4}.$$

Now, let $\delta = \min_{t=1,2,3,4}\{\delta_t\}$. We have that $\|F_i - F_i'\| < \delta$ implies

$$\left|m_i(s) - m_i'(s)\right| \equiv \left|\int_0^1 \sigma_i(v)\, n_i(v|s)\, dG(v) - \int_0^1 \sigma_i'(v)\, n_i'(v|s)\, dG(v)\right| < \varepsilon.$$

That is, $m_i(s)$ is continuous on $(F_i)_{i=1,2,3}$. $\square$ ■

Lemma B2 proves the existence admission strategies that satisfy the local conditions. The proof that those strategies are mutual (global) best responses is analogous to that of the two college case. We briefly summarize it below:

Consider a college, say i. For given $\sigma_j(\cdot)$ and $\sigma_k(\cdot)$, let $\tilde{\sigma}_i(v) \in [0,1]$ be an arbitrary strategy for $v \in [0,1]$. Let $\tilde{\sigma}_i(v;t)$ be a variation of $\sigma_i(\cdot)$ such that for any $t \in [0,1]$,

$$\sigma_i(v;t) := t\tilde{\sigma}_i(v) + (1-t)\sigma_i(v).$$

Define i's payoff function in terms of $\sigma_i(v;t)$,

$$V(t) := \int_0^1 v\, \sigma_i(v;t)\, \overline{n}_i(v)\, dG(v) - \lambda \int_0^1 \max\left\{ \int_0^1 \sigma_i(v;t)\, n_i(v|s)\, dG(v) - \kappa, 0 \right\} ds.$$

Observe that $V(\cdot)$ is continuous and concave in t. Therefore, we have

$$\pi_i(\tilde{\sigma}_i) = V(1) \leq V(0) + V'(0) \leq V(0) = \pi_i(\sigma_i),$$

where the second inequality holds since

$$V'(0) = \int_0^1 [\tilde{\sigma}_i(v) - \sigma_i(v)] H_i(v, \sigma_j(v), \sigma_k(v))\, dG(v) \leq 0 \tag{B.1.7}$$

because if $H_i(v, \sigma_j(v), \sigma_k(v)) \geq 0$ for some v, then $\sigma_i(v) = 1 \geq \tilde{\sigma}_i(v)$; and if $H_i(v, \sigma_j(v), \sigma_k(v)) \leq 0$, then $\sigma_i(v) = 0 \leq \tilde{\sigma}_i(v)$; and $H_i(v, \sigma_j(v), \sigma_k(v)) = 0$ otherwise.