The Diffusion of Network Goods:
A Two-Sided Interaction Approach*

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Abstract: I propose a new model analyzing the diffusion of a good sold by a monopolist. The monopolist sells a subscription network good, a good where consumers pay a subscription price in each period to use the good. Consumers have relationships with one another in an underlying social network, and the consumer utility of the good increases as the number of neighbors using the same good increases. I consider the diffusion process where consumers optimize their choices based on the overall adoption in the last period, and the monopolist maximizes his over-time profits. I formulate the diffusion of the good as a dynamic programming problem. I define the steady state of the diffusion as the equilibrium of this market. I show that the equilibrium exists and is unique. I also show that the consumer belief about the adoption rate is stable in the equilibrium. I analyze how the equilibrium changes depending on the underlying network. Moreover, I measure the inefficiency of the market by comparing the socially optimal state and the equilibrium state.

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1. Introduction

1.1. Overview

Many economic decisions are influenced by social networks. The underlying social network structure could affect the individual's network externality, the benefit of a product that becomes larger as more of our acquaintances use the product. For instance, our desire to purchase a cell phone depends on how many of our acquaintances, such as family, friends, and peers, also have adopted telephones. Firms often take this network externality into consideration when optimizing product mobilization strategies. As such, the diffusion of a product arises as a consequence of a two-sided strategic interaction rather than one-sided: the interaction between consumers and firms.

Previous literature has not fully explored how the underlying social network affects firms' behavior. The seminal mathematical model of diffusion is known as the Bass Model after its author, and there have been many extensions (e.g. Jackson and Rogers (2007), López-Pintado (2006)), and Young (2009). The Bass model relates the adoption rates of imitators to the number of adopters in each time period, and explains the commonly observed S-shaped adoption pattern of products.[1] These models are mechanical in that agents do not behave strategically. Several papers attempt to design models where agents behave strategically (e.g. Granovetter (1978), Morris (2000), Jackson and Yariv (2007)). Although these papers are quite general and applicable to various economic situations, their attention is still limited to one-sided interactions, rather than two-sided interactions. Complementing the previous literature, I propose a two-sided interaction model on social networks for diffusion analysis by considering a monopolist who maximizes over-time profit and consumers who are *myopic best-responders*.

A standard graph representation of the underlying social network enables me to model a *direct physical network externality*. The direct physical network externality indicates that the network externality is generated through underlying direct connections, rather than indirect connections. For example, when I decide to buy a cell phone, I consider not a foreigner whom I would not contact, but my friends whom I would connect. Therefore, an agent's network externality depends only on her neighbors of the graph. In this paper, I mainly consider a *subscription network good*, which means that consumers need to pay subscription prices in every period to use the product.[2] Thus, each consumer

[1] Rogers (2003) provides a comprehensive summary on empirical studies on diffusion.
[2] Examples are telecommunication services and memberships in clubs or associations.

decides whether to subscribe to the product after comparing the subscription price and the network externality that she can derive. For the information structure of the consumers, I follow a standard assumption used in the network game literature in that only the degree distribution of the network is commonly known to the monopolist and consumers (e.g. Galeotti et al. (2010)).

In this setting, I analyze steady states of diffusion, rather than the shape of diffusion over time. I propose a new equilibrium concept, a price-degree pair where both the monopolist and consumers mutually maximize their interests. Assuming the monotone hazard rate property of the underlying degree distribution, my model provides three main results. The first result characterizes the steady state equilibrium of diffusion, showing the existence and uniqueness of the equilibrium. The second result shows that consumer beliefs in the equilibrium are robust to small perturbations, so the equilibrium is *stable*. This result excludes *tipping points* from the equilibrium. The third result does comparative statistics stating that downward hazard rate shifts increase the equilibrium pair, which implies that the monopolist's profit increases. The final result provides a quantitative result measuring the efficiency gap between the socially optimized surplus and the surplus in the equilibrium.

Another novelty of the model arises from using the *Euler equation* that characterizes the monopolist's optimal plan without concavity of the objective function. Here I work with a dynamic programming problem, where the objective function is the monopolist's over-time profits, and the state transition represents the consumer adoption behavior. Since the underlying degree distribution determines the demand function in each period, the profit function is not necessarily concave. Thus, the differentiability of the value function is not guaranteed, and it makes it difficult to apply the standard theory of dynamic programming. However, by applying a result in Milgrom and Segal (2002), I prove that the Euler equation of the problem still characterizes the monopolist's optimal strategy. Therefore, my model opens the possibility that one could apply social network theory to other topics in industrial organization theory, such as platform competition, without worrying about technical details.

1.2. Related Literature

The first related literature analyzes the diffusion process with nonstrategic interactions. The pioneer work is Bass 1969). The Bass model expresses the diffusion process by a simple ordinary differential equation and explains many diffusion curves that are S-shaped; diffusion begins slowly, then accelerates, and eventually converges. The model is also fairly tractable because the process is explained by a simple equation. There are several papers (e.g. Jackson and Rogers (2007), López-Pintado (2006), and Young (2009)) that follow this approach to study diffusion phenomena such as the spread of diseases. Going beyond the Bass's approach, I explicitly model the strategic interaction between the monopolist and consumers, and the diffusion process is derived as a result of their interaction. The shape of the diffusion process is expressed by a second order difference equation instead of an ordinary differential equation.

There are several papers on diffusion with strategic interactions (e.g. Granovetter (1978), Morris (2000), and Jackson and Yariv (2007)). In particular, Jackson and Yariv (2007) is the most closely related paper to my own. Jackson and Yariv assume that agents behave strategically, but the agents have only limited information about the degree distribution. In each period, the game may have multiple equilibria under complete information on the entire structure of the network (e.g. Bramoulle and Kranton (2007)). However, the game has a unique equilibrium in a Bayesian-Nash sense because of the private and incomplete information on the network architecture, the so-called *mean-field approximation*. For this reason, incomplete information is frequently assumed in various network models (e.g. Galeotti et al. (2010)), and my model also accepts the incomplete information structure to describe consumer optimizing behavior. However, unlike their model, I introduce a monopolist in the market who sets the prices over time. The monopolist changes prices over time to reach the unique equilibrium, which is stable. This result answers the equilibrium selection problem that is addressed in Jackson and Yariv (2007).

There are related works on the diffusion of network goods in industrial organization (IO) theory. Basic assumptions and definitions of network goods markets are provided in Katz and Shapiro (1985, 1986, 1994). The most closely related papers in terms of diffusion and monopolistic pricing are Dhebar and Oren (1985, 1986). Dhebar and Oren study the relationship between price and network size. Assuming *subscription network goods* and *myopic consumers*, the price increases to the *steady state level* as the network size increases. Cabral et al. (1999) show that the monopolist sets increasing prices of its *durable goods* as time goes by, assuming that consumers are *rational* but have incomplete information about the cost structure of the monopolist. Radner et al. (2010) consider the price pattern where the monopolist sets the target demand, and sets lower prices when current demand is below the target demand and higher prices when demand is higher than the target. However, they do not model consumers connection structure, which I do by applying social network theory. I also derive a similar result. When the network effect is small because of the low adoption rate, the monopolist sets a low price, but he increases the price as the network effect increases. In addition, I contribute to IO theory by providing a new modeling tool that reflects the consumers network structure; I find a new intersection of IO theory and social network theory.

The rest of the paper is organized as follows. The next section presents a model describing consumer behavior and illustrates the monopolist's problem. Section 3 introduces a new equilibrium concept that incorporates a two-sided interaction. Then I characterize and analyze the equilibrium of the market. In Section 4, I extend the model to a generaled utility function. Section 5 concludes thoughts. All proofs are in Appendix.

2. The Model

I start off this section by describing the underlying social network that determines the network effect of goods sold by a monopolist. Then, I model consumer behavior with an information structure. The monopolist's problem will be formulated at the end of the section.

2.1. The Consumers on the Social Network

I consider a society that consists of a set of consumers who have relationships with one another. This society is modeled by the network $G =< N, g >$ where N is a (finite) set of consumers and g is an adjacency matrix that represents the relationship between consumers. $N_i(g) = \{j \in N | g_{ij} = 1\}$ represents the set of neighbors of consumer i. The degree of consumer i is the cardinality of the set $N_i(g)$ and denoted by x_i. The fraction of consumers in the network with degree x is described by degree distribution $f(x)$ that satisfies $\sum_{x \in \mathbf{X}} f(x) = 1$ where $\mathbf{X}$ is the support of f.

The monopolist sets the price of a good that he sells. Consumers have two options: buying (1) or not buying (0) one unit of the product. Each consumer consumes one unit of the product or not. The good has a network effect. Specifically, if a consumer buys the product, then she can derive extra benefits when her neighbors also use the good. Among various different network effects in the industrial organization literature, I consider the direct physical network effect that Katz and Shapiro (1985) introduced. The direct physical network effect is a network effect that is generated only between consumers who have a link to each other in the underlying network. For example, a person's desire to purchase a cell phone depends on how many of her friends also use it; but he does not consider other users who have no relationship with him and will have no chance to call him.[3] To incorporate the direct physical network effect, I assume a consumer with degree x evaluates her willingness to pay (WTP) for the product by

$$u_x(a, m) = \begin{cases} \gamma + v(m) & \text{if } a = 1 \\ 0 & \text{if } a = 0 \end{cases}$$

where $m(\leq x)$ is the number of neighbors who also use the product. γ measures the value in isolation, and v, which is a function from $\mathbf{X}$ to $\mathbb{R}$, captures the network effect. Without further loss of generality, I normalize $\gamma = 0$, which means that consumers have no basic WTP for the product. I assume that $v(m) = m$, which means that there is a linear positive network effect. Thus, when a consumer with degree x has belief θ as the independent probability of each of her neighbors' adoption, then her expected utility becomes

$$\mathbb{E}[u_x(1, M)] = \sum_{m=0}^{x} \binom{x}{m} m\theta^m (1-\theta)^{x-m} = x\theta$$

because M, the number of neighbors using the product, is the binomial random variable with parameter (x, θ).

Assume that consumers believe that each of their neighbors use the product with the same probability $\theta > 0$. Specifically, when the posted price is p, a consumer with degree x buys the good only if her expected utility is greater than or equal to p.[4] Since the only heterogeneity of consumers is their degrees, every consumer with the same degree takes the same action because the population shares the same belief of θ. Thus, there is a degree x' such that all consumers having more than x' buy the product. This observation can be formulated as the following lemma.

Lemma 1. *Suppose that consumers share θ, the belief that each of their neighbors buy the product. Then there is a unique threshold degree $x' \in \mathbf{X} \cup \{-1\}$ such that any consumers having more than x' degree buy the good and the others do not.*

I extend the above setting to a diffusion model in infinite discrete time. I assume that the good

[3]One can find other types of network effects and corresponding examples in Katz and Shapiro (1985, 1994).
[4]For simplicity, I assume that consumers also buy the good when their expected utilities are the same as the price.

is a subscription network good so that consumers need to pay a subscription price to use the product in each period. In each time t, lemma 1 can be generalized. Suppose that consumers have the same positive probability θ_0 at time $t = 1$, which is exogenously given. Then, if the posted price is p_1, a consumer with degree x subscribes to the good only if her expected utility is higher than or equal to p_1. Hence, there is a threshold degree x_1 such that all consumers with more than x_1 degree subscribe. This argument holds in every period so long as consumers have the same subscription probability of their neighbors. I formulate this observation as the following lemma.

Lemma 2. *Suppose that consumers share θ_t, the belief that each of their neighbors subscribe to the good. Then there is a unique threshold degree $x'_t \in \mathbf{X} \cup \{-1\}$ such that any consumers having more than x'_t degree subscribe to the good and the others do not.*

For consumer diffusion behavior, I mainly follow the modeling technique introduced in Jackson and Yariv (2007) where consumers myopically best respond to adoption of the population in the previous period. For the information structure of the model, I use a standard modeling technique (e.g. Jackson and Yariv (2007) and Galeotti et al. (2010)) in the network theory literature.[5] Specifically, consumers are partially informed about the network structure. The degree distribution of the underlying network $f(\cdot)$ is common knowledge across the population. However, each consumer knows her own degree without knowing her neighbors' degree. Let $\widetilde{f}(x) = xf(x)/<x>$, where $<x> = \mathbb{E}_F[x]$, be the degree distribution conditional on a consumer with degree x being at the end of a randomly chosen link in the network.[6] Then, $1 - \widetilde{F}(x)$ measures the probability that a consumer's neighbor has more than degree x where $\widetilde{F} : \mathbf{X} \to [0, 1]$ is the cumulative distribution function of $\widetilde{f}(\cdot)$.

Assuming x_t is the threshold degree at time t, $S(x_t) = \{x \in \mathbf{X} | x > x_t\}$ represents the set of subscribers, and its size is measured by $1 - F(x_t)$.[7] Consumers update the subscription probability in the next period by $\theta_{t+1} = 1 - \widetilde{F}(x_t)$. The intuition of this updating rule is that consumers can obtain the information about the adoption rate from media, for example. Consumers do not anticipate their neighbors adoption in the previous period because they are myopic. Hence, a consumer having degree x subscribes to the good at time $t+1$ if $x(1 - \widetilde{F}(x_t)) \geq p_{t+1}$. Applying lemma 2, the threshold degree

[5]This information structure with myopic best response is called the mean-field approximation in the network theory literature.

[6]The derivation is simple as one can guess that the probability should be proportional to the size of degree and the summation has to be normalized to sum up to 1 for being a density function. The detailed description can be found in Jackson (2008).

[7]Similarly, $\mathbf{X} - S(x_t)$ denotes the set of non-subscribers.

at time $t+1$ is determined.

2.2. The Demand Structure and the Monopolist's Problem

I approximate the degree distribution by an atomless continuous probability distribution.[8] For notational convenience, I denote F to be the approximated (cumulative) degree distribution with the support $\mathbf{X} = [0, M] \subseteq \mathbb{R}_+$ and corresponding $\widetilde{F}$. I assume F and $\widetilde{F}$ satisfy the strictly increasing hazard rate property (SIHRP), and they are continuously differentiable at least twice.[9]

I define function $g : \mathbf{X} \to \mathbb{R}$ by $g(x) = x\left(1 - \widetilde{F}(x)\right)$. This function measures the WTP of the non-subscribers with degree x when x is the threshold degree in the previous period. Similarly, define a function $h : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ by $h(x, x') = x\left(1 - \widetilde{F}(x')\right)$ that measures the WTP of consumers having x neighbors when x' is the threshold degree in the previous period. By the SIHRP, g has a single peak[10] and $g(0) = 0$.

Assume that p_t be the subscription price at time t and x_{t-1} be the threshold degree at time $t-1$. Then. when $p_t \geq 0$, x_t satisfies the equation $h(x_t, x_{t-1}) = p_t$ since the degree distribution is continuous.[11] Hence, the demand at time t is the set of subscribers, and its size becomes $1 - F(x_t)$. The demand $D(p_t, x_{t-1})$ is defined as $1 - F(x_t)$, which depends on p_t and x_{t-1}. Figure 1 illustrates the demand structure when the degree distribution is uniform on $[0, M]$.[12] In figure 1, where the threshold degree x_{t-1} is given, consumers' WTP is linear in degree. If the monopolist sets p_t in between 0 and $g(x_{t-1})$, it determines x_t, and the demand size becomes $D(p_t, x_{t-1}) = 1 - F(x_t) = 1 - Mp_t/(M^2 - x_{t-1}^2)$. $D(p_t, x_{t-1})$ is linear in p_t and decreases in x_{t-1}. Thus, the network effect affects the size of the demand; the more consumers adopt, the higher the demand size will be.[13] To describe the evolution of the subscription rate, I follow the definition in Dhebar and Oren (1985) that the subscription network shrinks (expands) if $S(x_{t+1}) \subsetneq (\supsetneq) S(x_t)$ and is in a steady state if $S(x_{t+1}) = S(x_t)$ at time t.

Given the demand and the diffusion process of consumers, I can formulate the monopolist's problem

[8] For instance, when the degree distribution is Erdös-Rényi(n, p), then the truncated normal distribution with the parameter (λ, λ), where $\lambda = np$, approximates the degree distribution of the network for large enough n. This comes from the fact that the degree distribution of a Erdös-Rényi graph is approximated by a Poisson discrete probability distribution, which is approximated by the normal distribution with the same mean and variance again.

[9] Mathematically, probability density function f satisfies the SIHRP if $\frac{\partial}{\partial x}\left(f(x)/(1 - F(x))\right) > 0$, and F and $\widetilde{F}$ are in $C^2([0, M])$.

[10] To see this, observe that $g'(x) = (1 - \widetilde{F}(x)) - x\widetilde{f}(x) = 0$ has a unique solution because $\widetilde{f}(x)/(1 - \widetilde{F}(x))$ strictly increases.

[11] x_t exists and is unique since h strictly increases in x.

[12] The uniform degree distribution satisfies the SIHRP assumption, and $g(x)$ has a single peak.

[13] Mathematically, $h_2(x, x_{t-1}) < 0$.

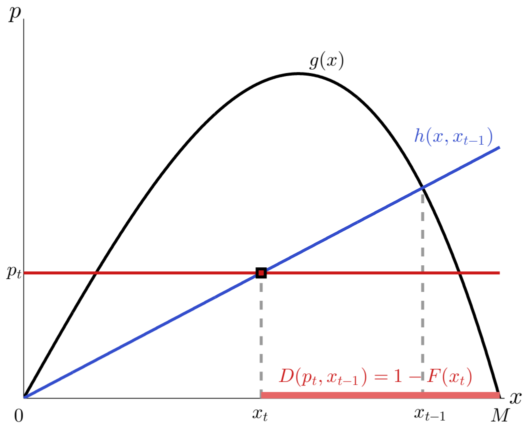

Figure 1: The Demand Structure

as a dynamic programming problem. For the monopolist's production technology, I assume that the monopolist pays unit production costs of $c(\geq \gamma)$, and let $c = 0$ for simplicity. The monopolist's profit at time t is represented by $p_t\,(1 - F(x_t))$. The monopolist tries to maximize his over-time profit with respect to the given consumer adoption behavior, which is written as the following.

$$\begin{aligned} \underset{\{p_t\}_{t=1}^{\infty}}{\text{maximize}} \quad & \sum_{t=1}^{\infty} \beta^{t-1} p_t\,(1 - F\,(x_t)) \\ \text{where} \quad & x_t = \frac{p_t}{1 - \widetilde{F}(x_{t-1})} \\ & x_0 > 0 \text{ is given.} \end{aligned} \tag{1}$$

3. Equilibrium Structure

In this section, I define the two-sided steady state equilibrium that incorporates maximizing behavior of both the monopolist and consumers. Then, I will prove its existence and uniqueness. I will discuss relationships between statistical properties of the degree distribution and the monotonicity of the equilibrium. A welfare analysis will follow.

3.1. The Equilibrium Concept

The main concern of this paper is to model a two-sided interaction in social networks for diffusion analysis. Since I consider a monopolist and myopic consumers, I need a new equilibrium concept that allows both sides to maximize their objectives. Consumers myopically best respond based on their belief in the previous adoption rate of the product, and the monopolist maximizes his profit. Thus, for the market to be in an equilibrium state, the subscription price has to return the monopolist's maximized profit, and the corresponding subscription network has to be in a steady state for the price. The monopolist sets the price and the consumers decide whether to buy; thus a price and adoption rate pair determines a state of the market. I define the two-sided steady state equilibrium as the following.

Definition (The Two-Sided Steady State Equilibrium)**.** *A price-degree pair* (p^*, x^*) *is a two-sided steady state equilibrium if*

1) p^* *maximizes the monopoly profit with respect to* x^* *(monopolist's first order condition).*

2) x^* *is a steady state with respect to* p^* *(no diffusion condition).*

Figure 2 illustrates situations when a price-state pair can be a two-sided steady state equilibrium and when it cannot be. In the left graph in figure 2, $(\hat{p}, \hat{x})$ does not satisfy the second condition of

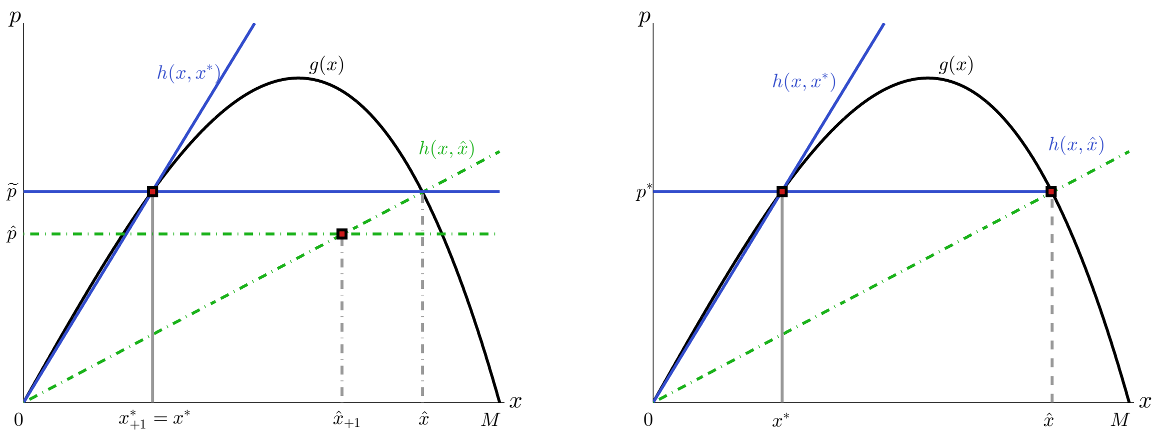

Figure 2: Equilibrium States vs Non-equilibrium States

the equilibrium definition because $\hat{x}$ is not a steady state with respect to $\hat{p}$. When the monopolist sets a lower price than $g(\hat{x})$, additional consumers ($[\hat{x}_+, \hat{x}]$ in the figure) will subscribe, and the network expands. This corresponds to the situation where the telecommunication service fee is low, and some consumers with higher WTP than the price want to join the service. However, x^* is a steady state because the price conforms to the WTP of the marginal consumers of degree x^* so that (p^*, x^*) can be an equilibrium. In the right graph in figure 2, $(p^*, \hat{x})$ cannot be an equilibrium because p^* does not maximize profit with respect to $\hat{x}$. If the monopolist sets a slightly lower price than $\hat{p}$, the profit increases because the slope of $h(x,\hat{x})$ is strictly less than 1. Hence, $(\hat{p},\hat{x})$ fails to satisfy the first condition in the equilibrium definition. On the other hand, (p^*,x^*) can be an equilibrium because p^* can maximize profit with respect to x^* depending on other characteristics of the monopolist and the underlying network structure. Therefore, the two-sided equilibrium is the price-degree pair that agrees with the marginal consumers' taste and returns the highest profit to the monopolist.

3.2. Properties of the Equilibrium

3.2.1. Existence and Uniqueness of the Equilibrium

I now examine the existence and uniqueness of the equilibrium. Since the equilibrium price should return the highest profit to the monopolist, an equilibrium state x^* solves the Euler equation of the dynamic programming problem (1). These two properties are represented by two equations $(1-\widetilde{F}(x^*))(1-F(x^*)) + x^*(1-\widetilde{F}(x^*))(-f(x^*)) + \beta v'(x^*) = 0$ and $v'(x^*) = x^*(-\widetilde{f}(x^*))(1-F(x^*))$, which respectively correspond to the first order condition and the envelope theorem of the dynamic programming problem where v is the value function. In addition, the equilibrium state has to be a steady state so that $p^* = g(x^*)$ has to be satisfied. By combining all these equations, characterization of the equilibrium follows.

Theorem 1. *A pair (p^*,x^*) is a two-sided steady state equilibrium if and only if $p^* = g(x^*)$ where x^* satisfies*

$$\frac{1}{x^*} = \frac{f(x^*)}{1-F(x^*)} + \beta\frac{\tilde{f}(x^*)}{1-\widetilde{F}(x^*)}.$$

Moreover, the equilibrium is unique.

The equilibrium characterization is related to the hazard rate of the underlying distribution rather than to other statistical properties. To see the reason for this, consider the case where $\beta = 0$, so that the monopolist is myopic. The monopolist maximizes his profit when he chooses a price that matches the marginal revenue to the marginal cost, which is zero in the model. Since the hazard rate of f strictly increases, the marginal revenue $MR(x,y) = (1-\tilde{F}(x))(1-F(y)) + y(1-\tilde{F}(x))(-f(y))$ strictly decreases in y for a given x.[14] Therefore, there is a unique state y that returns the highest revenue so that the equilibrium condition is obtained by solving $1/x^* = f(x^*)/(1-F(x^*))$ and $p = g(x^*)$. When the monopolist is not myopic ($\beta > 0$), he cares about not only the current profit, but also the profit

[14] Setting a price is equal to choosing a threshold degree, so I use y as the choice variable instead of the price.

that he can earn in the future. Thus, he tries to increase the size of the network effect to shift up consumer WTP because he can increase future profit by charging a high price later although it hurts his current profit. At the equilibrium, therefore, the monopolist sets a lower price and encourages more consumers to buy the product than when the monopolist is totally myopic. In addition, the monopolist will increase the network effect to the level where the marginal future profit is equal to the marginal current profit loss. The SIHRP of $\tilde{f}$ induces a unique optimal increase of network expansion. The additional adoption is captured by the second term of the characterization equation.

As a corollary of theorem 1, the monopolist increases the network size by encouraging more consumers to subscribe, since he cares about future profits more. This is formally stated by the following corollary.

Corollary 1. *$S(x^*)$ increases in β.*

One interpretation of β is that it is the probability that the monopolist will continue his monopoly status. Thus the corollary implies that when the monopolist is certain about he faces no incumbent in the market, the monopolist would expand the subscription network.

3.2.2. Stability of the Equilibrium

Two issues of the equilibrium are whether it is robust to small perturbations in consumer beliefs, and whether the equilibrium is stable. Consumers may have incorrect beliefs on the subscription rate. For example, the monopolist tries to exaggerate statistics to show a higher adoption rate than it actually is, and he charges higher prices later. To analyze stability, I accept the stability notion in Jackson and Yariv (2007) and modify it to create the following definition.

Definition (Stability and Tipping States)**.** *A state x' is stable with respect to a price p' if there exists $\varepsilon > 0$ such that for any $x'_0 \in (x' - \varepsilon, x' + \varepsilon)$, the threshold degree sequence $x'_t = p'/(1 - \widetilde{F}(x'_{t-1}))$ converges to x' as $t \to \infty$. Otherwise, x' is called unstable or a tipping point.*

Here is a proposition stating that the equilibrium state is stable.

Proposition 1. *The equilibrium state is stable.*

The stability of the equilibrium state can be illustrated in the left graph in figure 3. Suppose that consumer beliefs are perturbed so that consumers believe that only $S(x_0)$ consumers subscribe, although the price is still $p^* = g(x^*)$. Since consumers think fewer consumers subscribe to the product than the equilibrium, the demand curve shifts down to $h(x, x_0)$. Consumers in $[x_1, x_0]$ still have higher WTP than the price, and they subscribe to the product. Then, the network expands, and additional consumers $[x_2, x_1]$ join the network. This expanding-joining continues until the adoption rate becomes the equilibrium. An analogous process takes place when consumers think too many other consumers have subscribed. That is, when consumers think the adoption rate is higher than at the equilibrium, some customers cancel their subscriptions, because the price is higher than their WTP, and the network size shrinks. This continues until the adoption rate becomes the equilibrium.

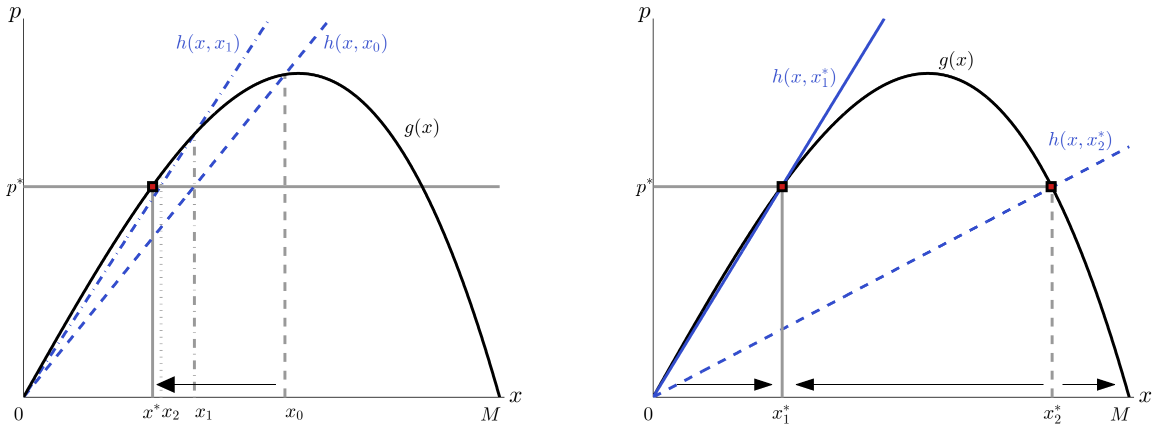

Figure 3: The Equilibrium is Stable

In the right graph in figure 3, two different degrees, x_1^* and x_2^*, exist that satisfy the self-fulfilling expectations of the network effect with respect to a fixed price p^*. If exactly $S(x_1^*)$ customers subscribe to the product, the price meets the threshold consumers' WTP so that x_1^* is a steady state. For the same reason, x_2^* is also a steady state. However, x_2^* is not stable, because for any small perturbation, the threshold degree sequence will not come back to x_2^*. If consumers believe the adoption rate higher than $S(x_2^*)$, the sequence converges to x_1^*; if consumers believe it should be lower, then the sequence converges to M. Moreover, x_2^* cannot be the equilibrium, because it does not maximize the monopolist's profit; if the monopolist sets a slightly lower price than p^* and then charges p^* from the next period on, he will increase his profits. Therefore, no unstable state can be the equilibrium.

3.2.3. Comparative Statics

I now analyze how the equilibrium changes depending on the underlying network structure. I consider a family of distributions $\{f(\cdot,\theta)\}_{\theta\in\Theta}$ indexed by the parameter θ in an ordered set Θ. I assume that $\{f(\cdot,\theta)\}_{\theta\in\Theta}$ has the strict monotone likelihood ratio property (SMLRP). [15] The SMLRP says that the degree is positively related to θ. If $\{f(\cdot,\theta)\}_{\theta\in\Theta}$ has the SMLRP, then $\{\widetilde{f}(\cdot,\theta)\}_{\theta\in\Theta}$ also has the SMLRP.[16] Since both F and $\widetilde{F}$ have the SMLRP, they have the first order stochastic dominance relationships and decreasing monotone hazard rate relationships in θ.[17] With these observations, I have the following proposition.

Proposition 2. *Let $(p^*(\theta), x^*(\theta))$ be the equilibrium when the underlying degree distribution is $f(\cdot,\theta)$. Then, $(p^*(\theta), x^*(\theta))$ strictly increases in θ.*

To see the intuition of proposition 2, consider the case where $\beta = 0$. For the monopolist, his marginal revenue, $MR(x, y, \theta) = (1 - \tilde{F}(x, \theta))(1 - F(y, \theta)) + y(1 - \tilde{F}(x, \theta))(-f(y, \theta))$, shifts up as θ changes from θ_0 to θ_1 since $f(\cdot, \theta_1)$ has more people with high degrees than $f(\cdot, \theta_0)$. To maximize his profit, the monopolist chooses a higher degree under θ_1 than under θ_0. For consumers, the g function also shifts up as θ increases from θ_0 to θ_1 because $\widetilde{f}(\cdot, \theta_1)$ has the first order stochastic dominance. Thus, the equilibrium price increases as θ increase. If $\beta > 0$, the monopolist cares about both the current profit and the profits that he can earn in the future. He would like to increase the size of the network effect to shift up consumer WTP as I discussed in section 3.2.1. For the same threshold degree, consumers believe that each of their neighbors have adopted the product with higher probability under $f(\cdot, \theta_1)$ because $\widetilde{f}(\cdot, \theta_1)$ has the first order stochastic dominance. When the monopolist choose a threshold degree, then the network effect expands more under $f(\cdot, \theta_1)$ than under $f(\cdot, \theta_0)$. In the equilibrium, therefore, the degree increases as θ increases, so the price also increases.

The SMLRP implies both the first order stochastic dominance and the monotone hazard rate. The former shifts up consumers' utility, and the latter shifts up the monopolist's marginal revenue. With these observations, the following corollary arises.

[15] Formally, $f(x_1,\theta_1)/f(x_1,\theta_0) > f(x_0,\theta_1)/f(x_0,\theta_0)$ for $\theta_1 > \theta_0$ and $x_1 > x_0$.

[16] This directly follows from the calculation:

$$\frac{\widetilde{f}(x_1,\theta_1)}{\widetilde{f}(x_1,\theta_0)} = \frac{x_1 f(x_1,\theta_1)/\mu(\theta_1)}{x_1 f(x_1,\theta_0)/\mu(\theta_0)} > \frac{x_0 f(x_0,\theta_1)/\mu(\theta_1)}{x_0 f(x_0,\theta_0)/\mu(\theta_0)} = \frac{\widetilde{f}(x_0,\theta_1)}{\widetilde{f}(x_0,\theta_0)}$$

where $\mu(\theta_i) = \int_{\mathbf{X}} x\, dF(x,\theta_i)$.

[17] Formally, for instance, $\widetilde{F}$ has the properties of $1-\widetilde{F}(x,\theta_1) > 1-\widetilde{F}(x,\theta_0)$ and $\widetilde{f}(x,\theta_1)/(1-\widetilde{F}(x,\theta_1)) < \widetilde{f}(x,\theta_0)/(1-\widetilde{F}(x,\theta_1))$ for all $x \in \mathbf{X}$.

Corollary 2. *The monopoly profit strictly increases in θ.*

As θ increases, the monopolist's profits increases if he chooses the same degrees on the optimal path. Therefore, the monopolist's equilibrium profit also increases as θ increases.

3.2.4. Consumer Welfare Loss

My model measures the consumer welfare loss from the monopoly. Since I assume $c = 0$, the socially efficient price-degree pair is $(p^S, x^S) = (0, 0)$; the social optimum is achieved when everyone subscribes to the product freely. With this, I measure the consumer welfare loss as the following proposition.

Proposition 3. *The consumer welfare loss is measured by*

$$\mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \leq x^*](1 - \widetilde{F}(x^*)) + \mathbb{E}_F[X]\widetilde{F}(x^*) \tag{2}$$

The proposition is presented by figure 4. When the monopolist sells the product, there are three different sources of the welfare loss. One loss is generated from the consumers who subscribe to the product in the equilibrium. The amount of loss from this corresponds to the region I in figure 4 and the first term in equation (2). If consumers with degrees in $[x^*, M]$ were to subscribe to the product freely, they would derive additional utility p^*. I measure the sum of these losses by multiplying the equilibrium price and the average degree of those consumers. Another loss is generated from the consumers who are excluded from the subscription network in the equilibrium. The amount of loss from this corresponds to the region II in figure 4 and the second term in equation (2). If consumers with degrees in $[x^S, x^*]$ were to subscribe to the product, they would derive the network effect of $1 - \widetilde{F}(x^*)$ from each of their neighbors. I can measure the sum of these losses by multiplying the lost network effect and the average degree of those consumers. The other loss is created from the whole population's welfare loss because the subscription network is not fully grown. The lost amount is represented by the region III in figure 4 and the third term in equation (2). If all consumers join the network, then every consumer derives the additional network effect of $\widetilde{F}(x^*)$. I can measure the sum of these welfare losses by multiplying the lost network effect and the average degree of the whole population. Therefore, the total welfare loss is represented by the sum of these two different losses.

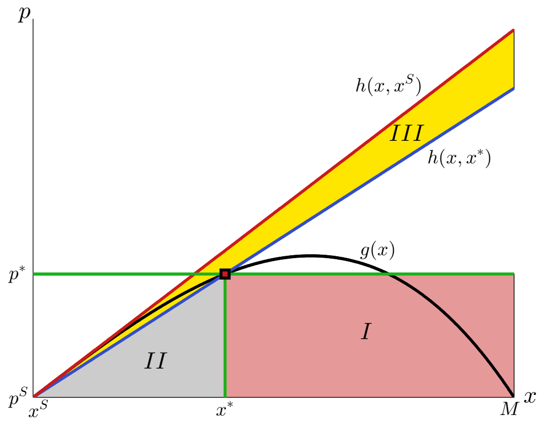

Figure 4: The Welfare Loss

4. Robustness

So far I have assumed that the utility function is linear in degree. However, the linearity of the expected utility function in degree would not be reasonable because it is equivalent to assume the constant marginal utility in degree. For instance, consumers may have time constraints so that they would not be able to interact with their friends fully, and the marginal utility from an additional neighbor decreases as the number of neighbors increases. For this reason, I consider a generalized expected utility function which has the form of $v(x)\theta$ where $v(\cdot)$ is a strictly increasing concave function with $v(0) = 0$. For the concavity of $v(\cdot)$, the Euler equation still characterizes the monopolist's optimal plan, so a similar equilibrium characterization result arises as the following.

Proposition 4. *Suppose that the consumers' expected utility function has the form of $v(x)\theta$ where $v(\cdot)$ is a strictly increasing concave function. Then, a unique equilibrium (p^*, x^*) exists. The equilibrium is characterized by $p^* = g(x^*)$ where x^* satisfies*

$$\frac{v'(x^*)}{v(x^*)} = \frac{f(x^*)}{1 - F(x^*)} + \beta \frac{\tilde{f}(x^*)}{1 - \widetilde{F}(x^*)}.$$

I consider $v(x) = x^\alpha$ with $\alpha \in (0, 1)$ as an example of the utility function satisfying the desired conditions. If the utility function has the form of $v(x) = x^\alpha$ with $\alpha \in (0, 1)$, the equilibrium characterization equation becomes that $p = g(x^*)$ where x^* satisfies

$$\frac{\alpha}{x^*} = \frac{f(x^*)}{1 - F(x^*)} + \beta \frac{\tilde{f}(x^*)}{1 - \widetilde{F}(x^*)} \tag{3}$$

from proposition 4. With this characterization, I analyze how the adoption rate in the equilibrium changes depending on the parameter α. I interpret the coefficient α as the elasticity of the network effect in degrees; as α decreases, the rate of marginal increase of the network externality slows down. I observe that the left hand side of the equation (3) increases in α. Therefore, fixing the underlying distribution, the product adoption rate increases as α decreases in the equilibrium.

5. Conclusion

This paper studies the equilibrium properties of diffusion of a network good. The diffusion arises as a sequence of equilibrium behaviors, where both a monopolist and consumers mutually maximize their interests. The equilibrium is defined as a steady state of the diffusion. The underlying network architecture affects consumer behavior, which determines demands. Since the monopolist optimizes his profit depending on demands, the underlying network structure also changes the monopolist's optimal plan. Assuming an increasing hazard rate of the degree distribution of the underlying network, I show that there exists a unique equilibrium.

In the equilibrium, consumer beliefs are robust to small perturbations, so the equilibrium is stable. This is because the equilibrium price is sufficiently low, so enough consumers subscribe to the product even with the underestimated network externality. Downward hazard rate shifts increase both the equilibrium price and degree, so the monopolist's profit shifts up. In addition, my model measures the efficiency gap between the socially optimized surplus and the equilibrium one. The inefficiency arises from the monopoly not only because some consumers are excluded from the subscription network, but also because the network externality does not fully grow in the equilibrium. The model is robust to more generalized consumer utility forms.

This paper is an opening work to study how social networks affect both consumer and firm behaviors. In particular, my study is important because the study provides a microeconomic foundation of utility functions of network goods, which has not been fully understood in industrial organization theory alone. My model considers only one firm, but the model is generally applicable to other related topics such as platform competition.

Appendix

1.1. Proofs of Section 2.1

Lemma 1. *Suppose that consumers share θ, the belief that each of their neighbors buy the product. Then there is a unique threshold degree $x' \in \mathbf{X} \cup \{-1\}$ such that any consumers having more than x' degree buy the good and the others do not.*

Proof. Let θ be a common consumers' belief and p be a posted price. Define $S(\theta, p) = \{x \in X | x\theta \geq p\}$. Since $X \subseteq \mathbb{N}_0$, $S(\theta, p)$ includes its minimum element. Let $x' = \min S(\theta, p) - 1$. Then, $x' \in \mathbf{X} \cup \{-1\}$, and x' satisfies the desired property. $\square$

Lemma 2. *Suppose that consumers share θ_t, the belief that each of their neighbors subscribe to the product. Then there is a unique threshold degree $x'_t \in \mathbf{X} \cup \{-1\}$ such that any consumers having more than x'_t degree subscribe to the good and the others do not.*

Proof. Define $S(\theta_t, p_t) = \{x \in X | x\theta_t \geq p_t\}$. Then, $S(\theta_t, p_t)$ has its minimum element. Thus, $x' = \min S(\theta_t, p_t) - 1$ is the degree that satisfies the desired property. $\square$

1.2. Proofs of Section 3.2.1

Theorem 1. *A pair (p^*, x^*) is a two-sided steady state equilibrium if and only if $p^* = g(x^*)$ where x^* satisfies*

$$\frac{1}{x^*} = \frac{f(x^*)}{1 - F(x^*)} + \beta \frac{\tilde{f}(x^*)}{1 - \widetilde{F}(x^*)}. \tag{1.1}$$

Moreover, the equilibrium is unique.

Proof. I show that the monopolist's problem is well-specified as a dynamic programming problem, and I derive a value function that characterizes the monopolist's optimal plan. Then I characterize the equilibrium assuming the differentiability of the value function and show existence and uniqueness of the equilibrium. Lastly, I show that the value function is differentiable.

To begin with, using the one-to-one correspondence between p_t and x_t, I rewrite the monopolist's problem as

$$\underset{\{x_t\}_{t=1}^{t=\infty}}{\text{maximize}} \quad \sum_{t=1}^{\infty} \beta^{t-1} \Pi(x_{t-1}, x_t)$$

where $\Pi(x_{t-1}, x_t) = x_t(1 - \widetilde{F}(x_{t-1}))(1 - F(x_t))$ and x_0 is given.

I now have a lemma that restricts the range of optimal plans.

Lemma 3. *Let $y^* \in \mathbf{X}$ be the degree that satisfies $\Pi_2(x, y) = 0$ and $\{x_t^*(x_0)\}_{t=1}^{\infty}$ be an optimal plan. Then, $x_t^*(x_0) \in [0, y^*]$ for all $t \geq 1$.*

Proof. Suppose, by way of contradiction, that there exists $t' \geq 1$ such that $x_{t'}^*(x_0) > y^*$. Since $y^* \in \mathbf{X}$ uniquely solves $\Pi_2(x_{t'-1}^*(x_0), y) = 0$ by the SIHRP, $\Pi(x_{t'-1}^*(x_0), y)$ has its unique maximum at y^*. If $t' = 1$, I have

$$\begin{aligned}\sum_{t=1}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) &= \Pi(x_0, x_1^*(x_0)) + \beta\Pi(x_1^*(x_0), x_2^*(x_0)) + \sum_{t=3}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) \\ &< \Pi(x_0, y^*) + \beta\Pi(y^*, x_2^*(x_0)) + \sum_{t=3}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0))\end{aligned}$$

which violates optimality of $\{x_t^*(x_0)\}_{t=1}^{\infty}$. For $t' \geq 2$,

$$\begin{aligned}\sum_{t=1}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) &= \sum_{t=1}^{t'-1} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) + \beta^{t'-1}\Pi(x_{t'-1}^*(x_0), x_{t'}^*(x_0)) \\ &\quad + \beta^{t'}\Pi(x_{t'}^*(x_0), x_{t'+1}^*(x_0)) + \sum_{t=t'+2}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) \\ &< \sum_{t=1}^{t'-1} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0)) + \beta^{t'-1}\Pi(x_{t'-1}^*(x_0), y^*) \\ &\quad + \beta^{t'}\Pi(x_{t'}^*(x_0), y^*) + \sum_{t=t'+2}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*(x_0), x_t^*(x_0))\end{aligned}$$

which violates optimality of $\{x_t^*(x_0)\}_{t=1}^{\infty}$. Therefore, I prove the statement. □

By the previous lemma, if $\beta = 0$, the optimal plan is $x_t^*(x_0) = y^*$ for all $t \geq 1$, so the equilibrium characterization equation follows. To prove the theorem for the case where $\beta > 0$, I set the feasible action correspondence, $\Gamma : \mathbf{X} \to \mathbf{X}$, as a constant map, $\Gamma(x) = [0, y^*]$ for all x. Then, Γ is a nonempty, compact, and convex subset of $\mathbb{R}$. With these observations, I have the following facts.

- $\Gamma(x)$ is nonempty for any $x \in \mathbf{X}$.
- For any $x_0 \in \mathbf{X}$ and any sequence $\{x_t\}_{t=1}^{\infty}$, $\lim_{n\to\infty} \sum_{t=1}^{n} \beta^{t-1}\Pi(x_{t-1}, x_t)$ exists because $\Pi(x_{t-1}, x_t)$ is uniformly bounded and $\sum_{t=1}^{n} \beta^{t-1}\Pi(x_{t-1}, x_t)$ is increasing in n.
- X is a convex subset of $\mathbb{R}$, and Γ is nonempty, compact-valued, and continuous.

- The function $\Pi : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0, 1)$.

With these four facts, it is well known that there is a unique continuous function $v : \mathbf{X} \to \mathbb{R}$ that solves $v(x) = \max\{y \in \Gamma(x) | \Pi(x, y) + \beta v(y)\}$ (see Stokey and Lucas (1989, Theorem 4.6) for details). I call v the value function.

Suppose that the value function is differentiable. Then, if there is a steady state x^*, it has to be the case that $v'(y) = \Pi_2(x^*, y) + \beta v'(y) = 0$ and $v'(x^*) = \Pi_1(x^*, y)$ at $y = x^*$. The former condition is called the *first order condition* and the latter is called the *envelope theorem*. By plugging the original profit function form into the two conditions, I have

$$(1 - \widetilde{F}(x^*))(1 - F(x^*)) + x^*(1 - \widetilde{F}(x^*))(-f(x^*)) + \beta v'(x^*) = 0 \tag{1.2}$$

and

$$v'(x^*) = x^*(-\widetilde{f}(x^*))(1 - F(x^*)).$$

By plugging the expression of $v'(x^*)$ into equation (1.2), I have

$$(1 - \widetilde{F}(x^*))(1 - F(x^*)) = x^*(1 - \widetilde{F}(x^*))f(x^*) + \beta x^* \widetilde{f}(x^*)(1 - F(x^*)).$$

By dividing both sides by x^* and $(1 - \widetilde{F}(x^*))(1 - F(x^*))$, I derive the steady state characterization equation as

$$\frac{1}{x^*} = \frac{f(x^*)}{1 - F(x^*)} + \beta \frac{\tilde{f}(x^*)}{1 - \widetilde{F}(x^*)}.$$

Since I assume that both F and $\widetilde{F}$ satisfy the SIHRP, the solution of this equation exists and is unique.

To apply the Euler equation of the monopolist's problem, it suffices to show that the value function is differentiable at optimal choices. To show this, I apply Milgrom and Segal's result (2002, Theorem 3), which states the following.

Theorem (Milgrom and Segal)**.** *Suppose that the family of functions $\{f_t(x, \cdot)\}_{x \in X}$ is equidifferentiable at $t_0 \in [0, 1]$, that $\sup_{x \in X} |f_t(x, t_0)| < \infty$, and that $X^*(t) \neq \emptyset$ for all t. Then V is left- and right-hand* *differentiable at* t_0*. For any selection* $x^*(t) \in X^*(t)$*, the directional derivatives are*

$$V'(t_0+) = \lim_{t \to t_0+} f_t(x^*(t), t_0) \text{ for } t_o < 1,$$
$$V'(t_0-) = \lim_{t \to t_0-} f_t(x^*(t), t_0) \text{ for } t_o > 0.$$

V *is differentiable at* $t_0 \in (0, 1)$ *if and only if* $f_t(x^*(t), t_0)$ *is continuous in* t *at* $t = t_0$*.*

Applying this theorem to my model, I need to show that

(1) $\{\Pi_1(\cdot, y)\}_{y\in[0,y^*]}$ is equidifferentiable at x_0 and $x_t^*(x_0)$ for all $t \geq 1$

(2) $\sup_{y\in[0,y^*]} |\Pi_1(x, y)| < \infty$ for $x = x_0$ and $x_t^*(x_0)$ for all $t \geq 1$

(3) $G(x) = \{y|v(x) = \Pi(x, y) + v(y)\} \neq \emptyset$ for all x

(4) $\Pi_1(x, y^*(x))$ is continuous in x at $x = x_0$ and $x_t^*(x_0)$ for all $t \geq 1$

(5) $\{x_t^*(x_0)\}_{t=1}^{\infty} \subseteq (0, y^*)$ where $\{x_t^*(x_0)\}_{t=1}^{\infty}$ is an optimal path.

Among these five conditions to prove, nontrivial conditions are condition (1) and condition (5).

To show condition (1), it suffices to show that $\{\Pi_{11}(\cdot, y)\}_{y\in[0,y^*]}$ is equicontinuous in x. I calculate $\Pi_{11}(\cdot, y)$ as $y(-\widetilde{f}'(\cdot))(1-F(y))$. Thus $\{\Pi_{11}(\cdot, y)\}_{y\in[0,y^*]}$ is equicontinuous since $y \in [0, y^*]$ and $\widetilde{F} \in C^2$.

To show condition (5), I use mathematical induction. For $t = 1$, I calculate

$$\begin{aligned} & \frac{\partial}{\partial y}\left(\Pi(x_0, y) + \beta\Pi(y, x_2^*(x_0)) + \sum_{t=3}^{\infty} \beta^{t-1}\Pi(x_{t-1}^*, x_t^*(x_0))\right) \\ = \ & (1 - \widetilde{F}(x_0))(1 - F(y)) + y(1 - \widetilde{F}(x_0))(-f(y)) + \beta x_2^*(x_0)(-\widetilde{f}(y))(1 - F(x_2^*(x_0))). \end{aligned} \tag{1.3}$$

At $y = 0$, the equation (1.3) is

$$(1 - \widetilde{F}(x_0))1 + \beta x_2^*(x_0)(-\widetilde{f}(0))(1 - F(x_2^*(x_0)))$$

and the last term becomes zero since $\widetilde{f}(0) = 0$. Therefore, $x_1^*(x_0)$ is strictly greater than zero.

Assuming $x_t^*(x_0) > 0$ where $x_t^*(x_0)$ is an optimal choice at time t, I show that $x_{t+1}^*(x_0) > 0$ since

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\sum_{s=1}^{t}\beta^{t-1}\Pi(x_{t-1}^*, x_t^*) + \beta^t\Pi(x_t^*(x_0), y) + \beta^{t+1}\Pi(y, x_{t+1}^*(x_0)) + \sum_{s=t+3}^{\infty}\beta^{s-1}\Pi(x_{s-1}^*(x_0), x_s^*(x_0))\right)\\
&= \beta^t\Big((1-\widetilde{F}(x_t^*(x_0)))(1-F(y)) + y(-f(y))\Big) + \beta^{t+1}x_{t+2}^*(x_0)(-\widetilde{f}(y))(1-F(x_{t+2}^*(x_0))) \qquad (1.4)\\
&> 0
\end{aligned}$$

at $y = 0$.

Similarly, I also show that (1.3) and (1.4) are strictly negative at $y = y^*$ using mathematical induction again. Thus any optimal choice is strictly less than y^*. Therefore, any optimal path locates in the interior of $[0, y^*]$, and so v is differentiable at $x_t^*(x_0)$ for all $t \geq 1$. □

Corollary 1. *$S(x^*)$ increases in β.*

Proof. The right hand side of the equilibrium characterization equation increases as β increases. This implies that the equilibrium degree x^* decreases. Therefore, the equilibrium adoption rate increases as β increases. □

1.3. Proof of Section 3.2.2

Proposition 1. *The equilibrium state is stable.*

Proof. Since the hazard rate of $\widetilde{f}$ increases, the equilibrium state x^* is in $(0, y^*)$. Then, there is $\varepsilon > 0$ such that $(x' - \varepsilon, x' + \varepsilon) \subseteq (0, y^*)$. Let x_0' be in $(x^*, x^* + \varepsilon)$. Then, $x_1' < x_0'$ because $p^* = g(x^*) < g(x_0')$. Assuming $x_t' < x_{t-1}$, $x_{t+1}' < x_t'$ for the same reason. In addition, $x^* \leq x_1'$ since $x^* = x^*(1-\widetilde{F}(x^*))/(x-\widetilde{F}(x^*)) < x^*(1-\widetilde{F}(x^*))/(1-\widetilde{F}(x_0')) = x_1'$. For the same reason, $x^* < x_t'$ for all $t \geq 1$. Thus, x_t' is a strictly decreasing sequence bounded below by x^*, and so x_t' converges to x^*. Similarly, for x_0' in $(x^* - \varepsilon, x^*)$, x_t' strictly increases to x^*. Therefore, I prove the statement. □

1.4. Proofs of Section 3.2.3

Proposition 2. *Let $(p^*(\theta), x^*(\theta))$ be the equilibrium when the underlying degree distribution is $f(\cdot, \theta)$. Then, $(p^*(\theta), x^*(\theta))$ strictly increases in θ.*

Proof. The increase of $x^*(\theta)$ directly follows from the characterization of the equilibrium. To show that $p^*(\theta)$ increases in θ, observe that

$$\begin{aligned} p^*(\theta') &= x^*(\theta')(1-\widetilde{F}(\theta',\theta')) \\ &< x^*(\theta')(1-\widetilde{F}(\theta',\theta'')) \text{ since } \widetilde{F}(\cdot,\theta'') \text{ has the first order stochastic dominance} \\ &< x^*(\theta'')(1-\widetilde{F}(\theta'',\theta'')) \text{ since } g \text{ increases in } x \text{ and } y^*(\theta'') < y^*(\theta'') \\ &= p^*(\theta''). \end{aligned}$$

Therefore, I prove the statement. □

Corollary 2. *The monopoly profit strictly increases in* θ*.*

Proof. Suppose that $\theta'' > \theta'$. Then, since $F(\cdot,\theta'')$ has the first order stochastic dominance, the monopolist's profit by charging $p = x^*(\theta')(1-\widetilde{F}(x^*(\theta'),\theta''))$ generates the higher profit than $p^*(\theta') = x^*(\theta')(1-\widetilde{F}(x^*(\theta'),\theta'))$ under $F(\cdot,\theta')$. Therefore, the monopolist drives the equilibrium profit under $F(\cdot,\theta'')$ at least as he earns under $F(\cdot,\theta')$. □

1.5. Proof of Section 3.2.4

Proposition 3. *The consumer welfare loss is measured by*

$$\mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \leq x^*](1-F(x^*)) + \mathbb{E}_F[X]F(x^*)$$

Proof. The following calculation proves the proposition:

$$\begin{aligned} &\mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X] - \mathbb{E}_F[X(1-F(x^*))|X \geq x^*] \\ &= \mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \geq x^*] + \mathbb{E}_F[X \leq x^*] - \mathbb{E}_F[X|X \geq x^*](1-F(x^*)) \\ &= \mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \leq x^*] + \mathbb{E}_F[X|X \geq x^*]F(x^*) \\ &= \mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \leq x^*](1-F(x^*)+F(x^*)) + \mathbb{E}_F[X|X \geq x^*]F(x^*) \\ &= \mathbb{E}_F[X \geq x^*]p^* + \mathbb{E}_F[X \leq x^*](1-F(x^*)) + \mathbb{E}_F[X]F(x^*). \end{aligned}$$

□

1.6. Proof of Section 4

Proposition 4. *Suppose that the consumers' expected utility function has the form of* $v(x)\theta$ *where* $v(\cdot)$ *is a strictly increasing concave function. Then, a unique equilibrium* (p^*, x^*) *exists. The equilibrium is characterized by* $p^* = g(x^*)$ *where* x^* *solves*

$$\frac{v'(x^*)}{v(x^*)} = \frac{f(x^*)}{1 - F(x^*)} + \beta \frac{\tilde{f}(x^*)}{1 - \widetilde{F}(x^*)}.$$

Proof. Since $v(\cdot)$ is a strictly increasing concave function, the entire proof of theorem 1 still holds. Therefore, the proposition directly follows. □

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