OPTIMAL DYNAMIC MECHANISM DESIGN WITH DEADLINES

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Abstract. A dynamic mechanism design problem with multi-dimensional private information is studied. Identical objects are sold to buyers who arrive over a finite number of periods. Besides his privately known valuation, each buyer has a privately known deadline for buying. The seller wants to maximize revenue. First, we derive the optimal mechanism, neglecting the incentive constraint for the deadline, and find two effects that determine whether the constraint is fulfilled automatically or not. Sufficient conditions on the type distribution are given for either case. Next, we consider a model with one object, two periods, and two buyers for the case that the constraint cannot be neglected. Here, the optimal mechanism is implemented by a fixed price in period one and an asymmetric auction in period two. The asymmetry, which is introduced to prevent the first buyer from buying in period one when his deadline is two, leads to pooling at the top of the type space.
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1. Introduction

The auction and mechanism design literature typically considers static allocation problems.[1] In many cases of interest, however, buyers arrive over time and care about when they obtain an object. Typical examples are online auctions, the sale of flight tickets, hotel reservations, or the sale of houses. We model time preferences in the form of deadlines. A deadline can be viewed as an extreme form of time preferences, as in the case of a traveler who needs to buy tickets before a certain date, in order to be able to coordinate with other travel arrangements. Deadlines may also be imposed by third parties. Consider a company that needs to buy a good from a

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[1]See for example the textbooks of Krishna (2002) and Milgrom (2004).

seller in order to enter a contractual relationship with a third party. This could be a physical object, an option contract, a license, a patent, etc. It is conceivable that the third party sets a deadline, after which the contractual relationship is no longer available. Therefore, the good is worthless for the company if it is purchased after the deadline.

We analyze the problem of a monopolistic seller with full commitment power, who sells one or more units of a good. The seller wants to maximize revenue in a dynamic environment, where buyers have independent private values and unit demand, arrive over a finite number of periods, and do not discount future payoffs. Each buyer is characterized by his arrival time, his valuation, and his deadline. A buyer's deadline and valuation are private information. To focus on the deadline, we assume that arrival times are observable for the seller.

When there is no discounting, it is optimal to allocate to a buyer only if his deadline is reached. For this class of mechanisms, we characterize incentive compatibility by one-dimensional constraints for the deadline and the valuation, respectively. Building on this, the paper makes the following contributions.

First, we derive the *relaxed solution*, which ignores the incentive constraint for the deadline. We give sufficient conditions on the type distribution, under which the neglected constraint is automatically fulfilled or violated. First, positive affiliation of deadlines and valuations leads to violations of incentive compatibility. We speak of a *static pricing* effect to describe this phenomenon, as it also arises in static models with two-dimensional private information (cf. Section 1.1).[2]

Second, we identify a new *dynamic pricing effect*. The novel insight is that the incentive constraint for the deadline may be violated even if the deadline and the valuation of a buyer are independent. Violations occur if the inverse hazard rate of the valuation is convex. If the inverse hazard rate is concave or linear, the neglected constraint is fulfilled automatically. In the latter case, the relaxed solution also solves the general problem. In the relaxed problem, the seller faces the classical trade off between efficiency and rent extraction. The rent extraction motive, which is reflected by the inverse hazard rate (Myerson, 1981), interacts with the dynamic arrival of buyers. Because of new arrivals, the allocation decision at a later deadline is based on more information. We show that this leads to higher rent extraction for later deadlines if the inverse hazard rate is convex. Hence, buyers will mimic types with earlier deadlines to avoid rent extraction—the incentive constraint for

[2]Positive affiliation implies that the hazard rate of the valuation is decreasing in the deadline, which leads to the static pricing effect. If the hazard rate is increasing in the deadline, the static pricing effect works in the opposite direction and the relaxed solution fulfills the neglected constraint, as long as there is no dynamic pricing effect.

the deadline is violated. In the case of a concave inverse hazard rate, rent extraction is higher for earlier deadlines, and incentive compatibility is preserved, as it is not profitable to mimic later deadlines.

A third contribution is the solution for the case that the relaxed solution is not incentive compatible. We restrict the model to two periods, two buyers, and one object. The optimal mechanism has a simple structure. In the first period, the seller sets a fixed price. If buyer one does not accept, but indicates that he would be willing to purchase in the second period, the seller waits and conducts an auction that gathers both buyers. Otherwise, the object is offered to the second buyer for a fixed price. This mechanism is incentive compatible if buyer one never buys in the first period when his deadline is two. The seller has two instruments to ensure incentive compatibility. He can increase the price in the first period, and he can distort the auction format in the second period in favor of buyer one. Both instruments increase the expected payoff from the auction compared to the fixed price. We derive the optimal mechanism and show that the seller always uses both instruments.

The distortion leads to an asymmetric auction, even if both buyers have identically distributed valuations. Moreover, buyer one wins with certainty if his valuation is sufficiently high. In contrast to the relaxed solution, which fully separates buyers with different types for a large class of type distributions, full separation is not optimal if the incentive constraint for the deadline is binding. We derive a generalized virtual valuation for buyer one. As in the classic optimal auction (Myerson, 1981), a buyer wins if and only if his generalized virtual valuation is non-negative and higher than the generalized virtual valuation of his opponent. The generalized virtual valuation has an endogenous parameter that determines the magnitude of the distortion. A simple procedure to compute the optimal distortion is provided.

Finally, the paper makes a methodological contribution. Formally, if the relaxed solution is not incentive compatible, we have to solve an auction problem with a type-dependent participation constraint, since buyers have the "outside option" to buy before their deadlines. This is the first paper that solves such a problem. The solution resembles Jullien (2000), who studies a principal-agent problem with a type-dependent participation constraint. Methodologically, however, the auction problem requires a different approach because of discontinuous winning probabilities and the additional capacity constraint in the auction. We adopt an approach pioneered by Reid (1968), which seems to be new to the mechanism design literature, and use a characterization of the capacity constraint from Mierendorff (2009a).[3] Another

[3]The characterization is a generalization of Border's (1991) characterization for symmetric allocation rules. Matthews (1984) conjectured the result proved by Border (see also Chen, 1986). For an early application of a special case of the result see Maskin and Riley (1984).

problem arises because the usual hazard-rate assumption is not sufficient to guarantee monotonicity of the winning probability in the distorted auction. Reid also shows how a monotonicity constraint can be incorporated in a control problem.[4] Using this approach, we can show that Myerson's ironing procedure can be applied to the generalized virtual valuation to determine bunches in the valuation dimension.[5]

1.1. Related Literature

This paper contributes to the literature on dynamic mechanism design. Several authors have analyzed the sale of an inventory of objects to short-lived buyers, i.e. buyers who have to be served immediately.[6] This is also a standard assumption in the revenue management literature.[7] If buyers are long-lived, they can strategically choose the time of a purchase. Gallien (2006) analyzes an infinite horizon model with long-lived buyers who have a common discount factor. Under a condition on the interarrival time distribution, longevity does not affect the optimal mechanism: There is no recall in the optimal allocation rule; buyers are only served at arrival.[8] Board and Skrzypacz (2010) show that the no recall property fails if the time horizon is finite. In contrast to these papers, we abstract from discounting. Therefore, it is always optimal to delay sales until the deadline. Gershkov and Moldovanu (2009b,c, 2010) study dynamic mechanism design problems in which the seller learns about future buyers' type distributions from current buyers' types. This leads to informational externalities that can destroy incentive compatibility of the relaxed solution. We rule out this effect by assuming that types are independent.

In contrast to much of the literature, we assume that buyers have private information about time preferences in the form of deadlines. This has also been studied by Pai and Vohra (2008b), who focus on sufficient conditions for incentive compatibility of the relaxed solution. These authors also allow for private information about the arrival time and do not make restrictions on the number of periods or objects.[9] For the deadline, they suggest that the incentive constraint is slack if the hazard rate of the valuation is *sufficiently monotone* in the deadline. This roughly corresponds to

[4]Hellwig (2008) derived a version of Pontriyagin's maximum principle that allows for discontinuities and a monotonicity constraint. This cannot be applied, however, because of the capacity constraint.
[5]This is also new to the mechanism design literature. Myerson's approach is not applicable to control problems. Guesnerie and Laffont (1984) and Mussa and Rosen (1978) only give necessary conditions for bunches.
[6]See Das Varma and Vettas (2001); Vulcano et al. (2002); Gershkov and Moldovanu (2009a); Dizdar et al. (2010).
[7]See Elmaghraby and Keskinocak (2003) for a survey. McAfee and te Velde (2007) survey airline pricing. Su (2007) studies a model with long-lived buyers.
[8]The condition can also be found in the earlier literature on search with recall (Zuckerman, 1986).
[9]In the case of more than one object, we have to restrict the number of time periods to two.

the static pricing effect. The condition, however, cannot be applied directly to the the type distribution. For the arrival time, they show that simple monotonicity of the hazard rate in the arrival time is sufficient (cf. Section 7).

In a different strand of literature, Said (2008) considers a scheduling model with stochastic arrival and exit of bidders. Pavan et al. (2008) consider a very general dynamic mechanism design model with agents who receive one-dimensional private information in every period. Finally, there is a literature on efficient dynamic mechanism design. (See Parkes and Singh (2003), Bergemann and Välimäki (2010), and Athey and Segal (2007) for existence results; Mierendorff (2009b) for the construction of a simple payment rule). In the context of the present paper, if the goal of the seller is value-maximization, the dynamic pricing effect vanishes, because the seller is not concerned about rent extraction any more. This shows that the effect depends on both dynamic arrivals and the seller's desire to maximize revenue.

This paper is also related to a literature on static mechanism design with two-dimensional private information, in which the second dimension is for example a budget constraint, a capacity requirement, or a quality constraint.[10] Such models are tractable because the second dimension has a special structure:[11] First, deviations are only possible in one direction (e.g. only under-reports of the budget or of the deadline are possible). Second, the second dimension is a constraint that does not enter the utility function as long as it is satisfied. For example, the utility of a buyer is independent of his deadline as long as he gets a unit before the deadline. Except for Szalay (2009), who considers a principal-agent problem, this literature typically makes assumptions to rule out an adverse static pricing effect, and concentrates on the case that the relaxed solution is incentive compatible.

Organization of the Paper

Section 2 describes the model. Section 3 characterizes incentive compatibility. Section 4 states the seller's problem. Section 5 presents the relaxed solution and conditions for incentive compatibility, formal proofs are in Appendix A. Section 6 presents the general solution for the binding case. The formal derivation is in Appendix B. Section 7 concludes. Some proofs are relegated to the Supplementary Appendix.

[10]See Beaudry et al. (2009) for an analysis of optimal taxation; Blackorby and Szalay (2008) and Szalay (2009) for regulation; Iyengar and Kumar (2008) and Dizdar et al. (2010) for auction models with capacitated bidders; Che and Gale (2000), Malakhov and Vohra (2005) and Pai and Vohra (2008a) for models with budget constrained buyers.

[11]The models of Rochet and Choné (1998) and Jehiel et al. (1999), in which all dimensions are symmetric, rarely have explicit solutions (see Armstrong, 1996, for an exception).

2. The Model

A seller wants to maximize the revenue from selling $K \in \mathbb{N}$ identical units of a good within $T \in \mathbb{N}$ time periods. The seller's valuation is normalized to zero. In each period, a random number of buyers $N_t \in \mathbb{N}_0$ arrives. The set of buyers who arrive in period t is denoted I_t and we write $I_{\leq t} = \bigcup_{\tau=1}^{t} I_\tau$ and $N_{\leq t} = |I_{\leq t}|$.

Each buyer is interested in buying at most one unit. A buyer $i \in I_t$ is characterized by his arrival time $a_i = t$, his valuation $v_i \in [0, \overline{v}]$, where $\overline{v} > 0$, and his deadline $d_i \in \{t, \dots, T\}$. The object cannot be sold to a buyer before his arrival time.

Utility is quasi-linear. If buyer i has to make a total payment of y_i, then his total payoff is $v_i - y_i$ if he gets (at least) one object in periods $a_i, \dots, d_i$, and $-y_i$ otherwise. Buyers are risk-neutral and maximize expected payoff. Neither the buyers nor the seller discount future payoffs.[12]

The numbers of arrivals in different periods are independently distributed. $\nu_{t,n}$ denotes the probability that n buyers arrive in period t. Deadline and valuation are jointly distributed for each buyer but independent for different buyers. Buyers with the same arrival period are ex-ante identical. For given arrival time a, the probability that the deadline of a buyer equals d is denoted $\rho_{a,d}$. Conditional on the deadline, the valuation has distribution function $F_a(v|d)$ and density $f_a(v|d)$.

Information realizes over time. In period t, the numbers of future buyers N_{t+1}, $\dots, N_T$, and their types are not known to anybody. In particular, the decision to sell a unit in period t cannot be based on this information. Upon arrival, each buyer privately observes his valuation and his deadline. In order to focus on the incentive issues of private information about deadlines, we assume that the seller observes arrivals.[13] $\nu_{t,n}$, $\rho_{a,d}$ and $F_a(.|d)$ are commonly known from the first period on.

We assume that for all a and all $d \geq a$, $f_a(v|d)$ is continuous in v and strictly positive for all $v \in [0, \overline{v}]$, continuously differentiable in v for $v \in (0, \overline{v})$, and that $f_1'(.|d)$ can be extended continuously to $[0, \overline{v}]$. To avoid additional technicalities, the following assumption is maintained throughout the paper.

Assumption 1. *For all $a \in \{1, \dots T\}$ and $d \in \{a, \dots T\}$, the* virtual valuation $J_a(v|d) := v - \frac{1 - F_a(v|d)}{f_a(v|d)}$ *is strictly increasing in v.*

The zero of $J_a(.|d)$ is denoted $v_a^0|d$ and v_T^0 if $a = d = T$. Finally, to exclude uninteresting cases, we assume that in each period, there is a positive probability of new arrivals ($\forall t : \nu_{t,0} < 1$).

[12]If only payments are discounted and all agents have a common discount factor, the results do not change. See Section 7 for a discussion of discounting.

[13]See Section 7 for a discussion of private information about arrival times.

2.1. Allocation Rules

In the most general formulation, a *state* $s_t = (H_t, \xi_{<t})$ consists of the history of buyer types $H_t = ((a_i, v_i, d_i))_{i \in I_{\leq t}}$, and the past allocation decisions $\xi_{<t} = (\xi_1, \ldots, \xi_{t-1})$, where $\xi_\tau \in \{0,1\}^{N_{\leq \tau}}$. $\xi_{\tau,i} = 1$ means that buyer i gets a unit in period τ. The history of buyer types, excluding the type of buyer i, is denoted $H_{t,-i}$. For a given state, the number of available units is denoted $k_t = K - \sum_{\tau=1}^{t-1} \sum_{i \in I_{\leq \tau}} \xi_{\tau,i}$.

Definition 1. (i) The set of *feasible allocations* in state $s_t = (H_t, \xi_{<t})$ is defined as

$$\Phi_t(s_t) = \left\{ \xi_t \in \{0,1\}^{N_{\leq t}} \middle| \sum_{i \in I_{\leq t}} \xi_{t,i} \leq k_t \right\}, \tag{F}$$

and the set of *allocations at the deadline* in state s_t is defined as

$$\tilde{\Phi}_t(s_t) = \{\xi_t \in \Phi_t(s_t) | \forall i \in I_{\leq t} : \xi_{t,i} = 0 \text{ if } d_i \neq t\}.$$

(ii) Let $x_t(\xi_t|s_t)$ denote the probability that allocation ξ_t is chosen in state s_t. An *allocation rule* $x = (x_1, \ldots, x_T)$ assigns a probability distribution over $\{0,1\}^{N_{\leq t}}$ to each state $s_t = (H_t, \xi_{<t})$, such that $x_t(\xi_t|s_t) = 0$ if $\xi_t \notin \Phi_t(s_t)$.

(iii) An allocation rule x *allocates only at the deadline* if $x_t(\xi_t|s_t) = 0$ for $\xi_t \notin \tilde{\Phi}_t(s_t)$.

(iv) An allocation rule is *symmetric* if for all t, all states s_t, all $\xi_t \in \Phi(s_t)$, and all $i, j \in I_{\leq t}$, such that $a_i = a_j$, $x_t(\xi_t|s_t) = x_t(\sigma_{i,j}(\xi_t)|\tilde{\sigma}_{i,j}(s_t))$.[14]

(v) A *payment rule* $y = (y_1, \ldots, y_T)$ assigns to each state $s_t = (H_t, \xi_{<t})$ and each $\xi_t \in \{0,1\}^{N_{\leq t}}$, a payment $y_{t,i}(s_t, \xi_t) \in \mathbb{R}$ for each $i \in I_{\leq t}$. A payment rule is *symmetric* if for all t, all s_t, all ξ_t and all $i, j \in I_{\leq t}$, such that $a_i = a_j$, $y_t(s_t, \xi_t) = \sigma_{i,j}(y_t(\tilde{\sigma}_{i,j}(s_t), \sigma_{i,j}(\xi_t)))$

2.2. Mechanisms

The seller's goal is to design a mechanism that has a Bayes-Nash-Equilibrium, which maximizes his expected revenue. We assume that the seller can commit ex-ante to a mechanism. In general, a mechanism can be any game form with T stages, such that only buyers from $I_{\leq t}$ are active in stage t. We assume that the mechanism designer can choose to conceal any information about the first t stages from the buyers that arrive in stages $t+1, \ldots, T$.[15]

[14] $\sigma_{i,j}$ is the permutation that interchanges the i^{th} and the j^{th} element of its argument and $\tilde{\sigma}_{i,j}(s_t) = (\sigma_{i,j}(H_t), (\sigma_{i,j}(\xi_1), \ldots, \sigma_{i,j}(\xi_{t-1})))$.

[15] This assumption yields an upper bound on the revenue that can be achieved. We will see that this bound can also be achieved in a periodic ex-post equilibrium, i.e. if buyers observe all information from past and current stages.

By the revelation principle, the seller can restrict attention to incentive compatible and individually rational direct mechanisms, in which no information is revealed. Since buyers who arrive in the same period are ex-ante identical, with out loss of generality, we can restrict attention to symmetric allocation and payment rules.

Definition 2. A *direct mechanism* consists of message spaces $S_1 = [0, \overline{v}] \times \{1, \ldots, T\}$, $\ldots$, $S_T = [0, \overline{v}] \times \{T\}$, and symmetric allocation and payment rules (x, y).

The (reported) state in period t can be constructed from the reports until period t, which yield H_t, and the past allocations $\xi_{<t}$.

The *interim winning probability* for period t, of a buyer $i \in I_a$ who reports (v', d'), if all other buyers (past, current and future) report their types truthfully, is given by (explicit expressions can be found in the Supplementary Appendix):

$$q_a^t(v', d') = \text{Prob}\{\xi_{t,i} = 1 | (a_i, v_i, d_i) = (a, v', d')\}.$$

The *interim expected payment* is given by

$$p_a(v', d') = E\left[\sum_{\tau=a}^{T} y_{\tau,i}(s_\tau, \xi_\tau) \middle| (a_i, v_i, d_i) = (a, v', d')\right],$$

where we aggregate payments from different periods. (q, p) is called the *reduced form* of (x, y). The *interim expected utility* from participating in a mechanism (x, y) with true type (v, d) and report (v', d') is given by

$$U_a(v, d, v', d') = \left[\sum_{\tau=a}^{d} q_a^\tau(v', d')\right] v - p_a(v', d'). \tag{2.1}$$

The expected utility from truth-telling is abbreviated $U_a(v, d) := U_a(v, d, v, d)$.

Definition 3. (i) A direct mechanism (x, y) is *(Bayesian) incentive compatible* if for all $a \in \{1, \ldots, T\}$, $v, v' \in [0, \overline{v}]$, and $d, d' \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq U_a(v, d, v', d'). \tag{IC}$$

(ii) A direct mechanism (x, y) is *individually rational* if for all $a \in \{1, \ldots, T\}$, $v \in [0, \overline{v}]$, and $d \in \{a, \ldots, T\}$,

$$U_a(v, d) \geq 0. \tag{IR}$$

3. Characterization of Incentive Compatibility

Since valuations are not discounted, the seller can restrict attention to direct mechanisms that allocate only at the deadline.

Lemma 1. *Let (x, y) be a direct mechanism that satisfies (IC) and (IR). Then, there exists an allocation rule $\hat{x}$ that allocates only at the deadline, such that the direct mechanism $(\hat{x}, y)$ also satisfies (IC) and (IR), and (x, y) and $(\hat{x}, y)$ yield the same expected revenue.*

Proof. The proof can be found in the Supplementary Appendix. □

In the rest of the paper, only mechanisms that allocate only at the deadline are considered and we write $q_a(v, d)$ instead of $q_a^d(v, d)$. With this restriction, buyers who were assigned units in the past have deadlines $d_i < t$, and their identities are not relevant for current and future allocation decisions. Therefore, we sometimes replace $\xi_{<t}$ by k_t in the state to simplify notation.

For the class of mechanisms that allocate only at the deadline, the two-dimensional incentive constraint (IC) is equivalent to two one-dimensional constraints.

Theorem 1. *Let (x, y) be a direct mechanism with reduced form (q, p) that allocates only at the deadline. Then (x, y) is incentive compatible if and only if for all $a \in \{1, \ldots, T\}$, all $d \in \{a, \ldots, T\}$, and all $v, v' \in [0, \overline{v}]$:*

$$v > v' \Rightarrow q_a(v, d) \geq q_a(v', d), \tag{M}$$

$$U_a(v, d) = U_a(0, d) + \int_0^v q_a(s, d) ds, \tag{PE}$$

$$U_a(v, d) \leq U_a(v, d + 1), \qquad \text{if } d < T, \tag{ICD$^{\mathrm{d}}$}$$

$$\text{and} \quad U_a(0, d) = U_a(0, d + 1), \qquad \text{if } d < T. \tag{ICD$^{\mathrm{u}}$}$$

Sketch of Proof. (M) and (PE) is the standard characterization of one-dimensional incentive compatibility for the valuation (Myerson, 1981). (ICD$^{\mathrm{d}}$) rules out under-reports of the deadline. Together with (M) and (PE), this also rules out simultaneous misreports of an earlier deadline $d' < d$ and a valuation $v' \neq v$. For mechanisms that allocate only at the deadline, the constraint takes this simple form because the utility of under-reporting the deadline is independent of the true deadline (cf. (2.1)):

$$d' \leq d \quad \Rightarrow \quad U_a(v, d, v', d') = U_a(v, d', v', d').$$

Incentive compatibility for the valuation implies that $U_a(v, d', v', d')$, and therefore also $U_a(v, d, v', d')$, is maximized by $v' = v$. For $v' = v$, (ICD$^{\mathrm{d}}$) rules out a downward deviation in the deadline. Therefore, simultaneous deviations in the deadline and the valuation are also ruled out. Necessity of (ICD$^{\mathrm{d}}$) is obvious.

Reporting $d' > d$ is only profitable if the mechanism pays a subsidy, i.e. if $p_a(v, d') < 0$. (PE) implies that subsidies are non-increasing in the valuation. Therefore, the highest subsidy (if any) is paid for $(0, d')$. By (PE), $v = 0$ is also the valuation for which over-reporting the deadline is most tempting. Hence, to rule out upward deviations, it suffices that $U_a(0,d) = -p_a(0,d) \geq -p_a(0,d') = U_a(0,d')$. Together with (ICD$^{\mathrm{d}}$) for $v = 0$, this is equivalent to (ICD$^{\mathrm{u}}$).[16] $\square$

Formally, the downward incentive constraint for the deadline resembles a type-dependent participation constraint. A buyer with arrival time a and deadline d has the "outside option" to report $d' \in \{a, \dots, d-1\}$. He only "participates" voluntarily with $d' = d$ if his payoff with $d' = d$ exceeds the payoff of his best "outside option."

4. The Seller's Problem

By the revelation principle and Lemma 1, the seller's problem is to choose an incentive compatible and individually rational direct mechanism that allocates only at the deadline, to maximize

$$\sum_{a=1}^{T} E[N_a]E[p_a(v,d)] = \sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\infty} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} p_a(v,d)f_a(v|d)dv\right].$$

Using (PE) to substitute the payment rule, integrating by parts and setting $U_a(0,d) = 0$ for all $a \in \{0,\dots,T\}$ and $d \in \{a,\dots,T\}$, the objective of the seller becomes

$$\sum_{a=1}^{T}\left[\left(\sum_{N_a=1}^{\infty} N_a\,\nu_{a,N_a}\right)\sum_{d=a}^{T}\rho_{a,d}\int_0^{\overline{v}} q_a(v,d)J_a(v|d)f_a(v|d)dv\right].$$

If we substitute $q_1(v,d)$, this can be rearranged to[17]

$$E_{s_1}\left[\sum_{\xi_1\in\tilde{\Phi}_1(s_1)} x_1(\xi_1|s_1)\left(\sum_{i\in I_1}\xi_{1,i}J_{a_i}(v_i|1) + E_{s_2}\left[\sum_{\xi_2\in\tilde{\Phi}_2(s_2)} x_2(\xi_2|s_2)\left(\sum_{i\in I_{\leq 2}}\xi_{2,i}J_{a_i}(v_i|2) + \right.\right.\right.\right.$$
$$\left.\left.\left.\left.\dots\ E_{s_T}\left[\sum_{\xi_T\in\tilde{\Phi}_T(s_T)} x_T(\xi_T|s_T)\sum_{i\in I_{\leq T}}\xi_{T,i}J_{a_i}(v_i|T)\,\middle|\, s_{T-1},\xi_{T-1}\right]\dots\right)\middle|\, s_1,\xi_1\right]\right)\right],$$

where E_{s_t} denotes the expectation with respect to s_t. It is more convenient to formulate the seller's problem as a recursive dynamic program $\mathcal{R}$:

$$V_T(s_T) := \max_{x_T}\sum_{\xi_T\in\tilde{\Phi}_T(s_T)} x_T(\xi_T|s_T)\left(\sum_{i\in I_{\leq T}}\xi_{T,i}J_{a_i}(v_i|T)\right), \qquad (\mathcal{R})$$

[16]If $v \in [\underline{v},\overline{v}]$ with $\underline{v} > 0$, then the upward incentive compatibility constraint for the deadline would be $q_a(\underline{v},d)\underline{v} - p_a(\underline{v},d) \geq -p_a(\underline{v},d+1)$. In this case, a subsidy could be used to separate buyers with different deadlines. One can show, however, that this instrument would not be used in the optimal mechanism, unless the allocation rule is sufficiently distorted. The reason is that the cost of a subsidy is of first order whereas the cost of distorting the allocation rule is of second order.

[17]See McAfee and McMillan (1987) for a similar derivation in a static model.

$$\forall t < T : V_t(s_t) := \max_{x_t} \sum_{\xi_t \in \tilde{\Phi}_t(s_t)} x_t(\xi_t|s_t) \left(\sum_{i \in I_{\leq t}} \xi_{t,i} J_{a_i}(v_i|t) + E_{s_{t+1}} \left[V_{t+1}(s_{t+1}) | s_t, \xi_t \right] \right),$$

where the reduced form of the optimal policy must satisfy (M) and (ICD$^{\mathrm{d}}$) with $U_a(v,d)$ given by (PE) and $U_a(0,d) \equiv 0$.

5. The Relaxed Solution

In order to derive conditions under which the constraint (ICD$^{\mathrm{d}}$) is binding, we first solve $\mathcal{R}$ subject to (M) only. This is the *relaxed problem* and corresponds to the case where deadlines are observed by the seller.

As in the classic optimal auction problem, Assumption 1 guarantees that (M) is slack for the optimal policy of the relaxed problem (Myerson, 1981). Therefore, we can ignore (M) in the derivation of the relaxed solution.

For a given state s_t, define $c^t_{(1)} \geq \ldots \geq c^t_{(K)}$ as the K highest virtual valuations among the buyers $i \in I_{\leq t}$ with deadlines $d_i = t$. Let $i^t_{(1)}, \ldots, i^t_{(K)}$ denote the identities of distinct buyers with these virtual valuations, i.e. for all $k = 1, \ldots, K$: $d_{i^t_{(k)}} = t$ and $J_{a_{i^t_{(k)}}}(v_{i^t_{(k)}}|t) = c^t_{(k)}$.[18] Furthermore, define $\Delta_{t+1}(s_t, k) = E_{s_{t+1}}[V_{t+1}(s_{t+1})|s_t, k_{t+1} = k] - E_{s_{t+1}}[V_{t+1}(s_{t+1})|s_t, k_{t+1} = k-1]$. $\Delta_{t+1}(s_t,k)$ is the marginal option value of retaining the k^{th} unit in period t. Note that the marginal option values are non-increasing in k. k^*_{t+1}, the optimal number of units that are retained for period $t+1$, is determined by the following conditions:

$$c^t_{(k_t - k^*_{t+1})} > \Delta_{t+1}(s_t, k^*_{t+1} + 1) \quad \text{if } k^*_{t+1} < k_t,$$
$$\text{and} \qquad c^t_{(k_t - k^*_{t+1} + 1)} \leq \Delta_{t+1}(s_t, k^*_{t+1}) \quad \text{if } k^*_{t+1} > 0.$$

The set of winning buyers in period t is given by

$$W^*_t(s_t) := \left\{ i^t_{(1)}, \ldots, i^t_{(k_t - k^*_{t+1})} \right\} \cap \left\{ i \in I_{\leq t} | J_{a_i}(v_i|t) \geq 0 \right\}.$$

The optimal policy for the relaxed problem is deterministic and given by

$$x^{\mathrm{rlx}}_t(\xi_t|s_t) = \begin{cases} 1 & \text{if } \xi_{t,i} = 1 \Leftrightarrow i \in W^*_t(s_t), \\ 0 & \text{otherwise.} \end{cases}$$

A buyer's type determines whether the buyer is in the set of winning bidders at his deadline, but it can also influence the number of units available at the deadline. Let $k^*_{a,d}(H_{d,-i}, (a,v,d), k_a)$ be the number of units available in period d, if buyer i arrives in period a, with type (a,v,d), and k_a units were available in the arrival period. Buyer i gets a unit if $i \in W^*_d((H_{d,-i}, (a,v,d)), k^*_{a,d}(H_{d,-i}, (a,v,d), k_a))$. Therefore, we

[18] All tie-breaking rules yield the same expected revenue.

define the *critical virtual valuation* of buyer i in state s_d for given k_a as

$$\zeta^i_{a,d}(H_d,k_a) := \inf\left\{\zeta \middle| i \in W^*_d((H_{d,-i},(a,J_a^{-1}(\zeta|d),d)),k^*_{a,d}(H_{d,-i},(a,J_a^{-1}(\zeta|d),d),k_a))\right\}.$$

With this definition, i gets a unit only if $J_{a_i}(v_i|d_i) \geq \zeta^i_{a,d}(H_d,k_a)$.[19]

The relaxed solution can be implemented by the following payment rule:

$$y_i^{\text{rlx}}(s_t,\xi_t) \quad = \quad \begin{cases} 0, & \text{if } \xi_{t,i}=0, \\ J_{a_i}^{-1}(\zeta^i_{a_i,d_i}(H_{d_i},k_a)|t), & \text{if } \xi_{t,i}=1. \end{cases}$$

With this payment rule, the payment of a losing buyer is zero and a winner pays the lowest valuation, with which he could have obtained a unit for given k_a and a given history of buyer arrivals in until period d. Thus, truth-telling is a weakly dominant strategy if the deadline is public and buyers only report their valuations.

Now we turn to the question whether the relaxed solution is incentive compatible if the deadline is privately known. In the relaxed solution, $U_a(0,d)=0$ for all $a \in \{1,\ldots,T\}$ and $d \in \{a,\ldots,T\}$. Hence, it suffices to check whether the expected payoffs for the payment rule y^{rlx} satisfy (ICD$^{\text{d}}$).

The following observation is crucial for the comparison of expected payoffs for different deadlines.

Lemma 2. *Let $K=1$ or $T \leq 2$. For all states s_a, and all $i \in I_a$ with deadline $d_i \geq a$, $\left(\zeta^i_{a,d}(H_d,k_a)\right)_{d=a,\ldots,d_i}$ is a martingale (with respect to $(H_d)_{d=a,\ldots,d_i}$): For all $d \in \{a+1,\ldots,d_i\}$,*

$$E_{H_d}\left[\zeta^i_{a,d}(H_d,k_a)|H_{d-1}\right] = \zeta^i_{a,d-1}(H_{d-1},k_a).$$

Furthermore, for all $d \in \{a,\ldots,d_i-1\}$,

$$\left[\zeta^i_{a,d}(H_d,k_a)\middle|s_a\right] \succ_{SSD} \left[\zeta^i_{a,d_i}(H_{d_i},k_a)\middle|s_a\right],$$

where $\succ_{SSD}$ denotes strict second-order stochastic dominance.

Proof. See Appendix A. □

Example 1. To illustrate the lemma, suppose that valuations and deadlines are independent and let $T=2$, $K=1$, $I_1=\{1,2\}$ and $I_2=\{3\}$. In this case, the critical virtual valuations of buyer $i=1$ for $d_1=1$ and $d_1=2$, respectively, are given by

$$\zeta^1_{1,1}(H_1,1) = \begin{cases} \max\left\{J_1(v_2),E_{v_3}\left[\max\left\{0,J_2(v_3)\right\}\right]\right\}, & \text{if } d_2=1, \\ E_{v_3}\left[\max\left\{0,J_1(v_2),J_2(v_3)\right\}\right], & \text{if } d_2=2, \end{cases}$$

[19] The converse is not necessarily true. If $J_{a_i}(v_i|d_i) = \zeta^i_{a,d}(H_d,k_a)$, the tie-breaking rule determines whether $i \in W^*_d((H_{d,-i},(a,v,d)),k^*_{a,d}(H_{d,-i},(a,v,d),k_a))$.

$$\zeta^1_{1,2}(H_2,1) = \begin{cases} \max\left\{z(J_1(v_2)), J_2(v_3)\right\}, & \text{if } d_2 = 1, \\ \max\left\{0, J_1(v_2), J_2(v_3)\right\}, & \text{if } d_2 = 2, \end{cases}$$
$$\text{where} \quad z(J_1(v_2)) = \min\left\{z \geq 0 \mid E_{v_3}\left[\max\left\{z, J_2(v_3)\right\}\right] \geq J_1(v_2)\right\}.$$

Let $d_1 = 2$ and $d_2 = 1$, and suppose that without buyer one, the object would be sold to buyer two in the first period ($J_1(v_2) > E_{v_3}[\max\{0, J_2(v_1)\}]$). Then, the object is retained in period one, if and only if buyer 1's virtual valuation is greater or equal than $z(J_1(v_2))$. In other words, buyer one must have a virtual valuation $J_1(v_1) \geq z(J_1(v_2))$ to overbid buyer two in the first period. Since $\max\left\{J_1(v_2), E_{v_3}\left[\max\left\{0, J_2(v_3)\right\}\right]\right\} = E_{v_3}\left[\max\left\{z(J_1(v_2)), J_2(v_3)\right\}\right]$, we have that $\zeta^1_{1,1}(H_1, k_1) = E_{v_3}[\zeta^1_{1,2}(H_2, k_1)|s_1]$ as stated in the lemma.

Buyer one always faces competition by both buyers two *and* three. Independently of his deadline, he competes directly with buyer two. Competition with buyer three is direct if $d_1 = 2$, and indirect through the option value of retaining the object if $d_1 = 1$. A later deadline has two effects, first it lowers the virtual valuation needed to overbid buyer two, because $z(J_1(v_2)) < J_1(v_2)$ and $J_1(v_2) < E_{v_3}\left[\max\left\{J_1(v_2), J_2(v_3)\right\}\right]$ if $v_2 < \overline{v}$. Second, a higher virtual valuation is needed to overbid buyer three whenever $J_2(v_3) > E_{v_3}\left[\max\left\{0, J_2(v_3)\right\}\right]$. The Lemma shows that the two effects cancel in expectation. If we interpret the critical virtual valuation as a measure of competition by other buyers, this shows that expected competition is independent of the reported deadline. Hence, differences in expected payoffs for different deadlines are not caused by a competition effect.

Second order stochastic dominance is obvious: conditional on s_1, $\zeta^1_1(s_1)$ is constant and therefore dominates $\zeta^1_2(s_2)$, which has the same expectation.

The following theorem gives sufficient conditions, under which the static and the dynamic pricing effects lead to incentive compatibility of the relaxed solution and violations of incentive compatibility, respectively.

Theorem 2. *Suppose that $K = 1$ or that $T \leq 2$. Then, in the relaxed solution,*

(i) ($\mathrm{ICD}^{\mathrm{d}}$) *is violated for type (a, v, d) if there exists $d' \in \{a, \dots, d-1\}$, such that*
   - *(a) $J_a(v|d') \geq J_a(v|d)$ for all $v \in [v^0_a|d', \overline{v}]$, and*
   - *(b) $J_a(v|d)$ or $J_a(v|d')$ is strictly concave as a function of v.*

   If $J_a(v|d') > J_a(v|d)$ for all $v \in [v^0_a|d, \overline{v})$, strict concavity can be replaced by weak concavity.

(ii) ($\mathrm{ICD}^{\mathrm{d}}$) *is satisfied for type (a, v, d) if for all $d' \in \{a, \dots, d-1\}$,*
   - *(a) $J_a(v|d') \leq J_a(v|d)$ for all $v \in [v^0_a|d, \overline{v}]$, and*
   - *(b) $J_a(v|d)$ or $J_a(v|d')$ is weakly convex as a function of v.*

Proof. See Appendix A. □

Condition (a) corresponds to the static pricing effect. There is a simple economic intuition. If we reformulate condition (a) in part (i), we get

$$\frac{f_a(v|d')}{1-F_a(v|d')} \geq \frac{f_a(v|d)}{1-F_a(v|d)}.$$

This means that the distribution of valuations for the later deadline d is *stronger* in the hazard rate order.[20] By revealing a later deadline, the buyer provides information about the valuation distribution to the seller. In the relaxed solution this information is fully exploited by the seller—a stronger distribution leads to higher rent extraction. Therefore, the buyer does not have an incentive to reveal a stronger valuation distribution—he will lie about his deadline if valuations and deadlines are positively related.

Condition (b) corresponds to the dynamic pricing effect. The following example illustrates that (a) it does not depend on stochastic dependencies, and (b), it depends on an interplay of dynamic arrival of new information (buyer arrival), and the seller's desire to maximize revenue.

Example 2. Suppose that $T = 2$ and $K = 1$. In each period, one buyer arrives. The first buyer can have deadline one or two. Assume that his valuation is independent of the deadline ($F_1(v_1|d) = F_1(v_1)$). The relaxed solution looks as follows: If $d_1 = 1$ the seller makes a take-it-or-leave-it offer of $y_1^{\mathrm{rlx}}(v_1, d_1 = 1)$ to buyer one. If the offer is rejected, he waits and makes an offer of $y_2^{\mathrm{rlx}}((v_1, d_1 = 1), (v_2))$ to buyer two. The optimal offer to buyer two is determined by $J_2(y_2^{\mathrm{rlx}}) = 0$, which implies that the expected revenue of waiting for period two equals $\zeta_{1,1} = y_2^{\mathrm{rlx}}(1 - F_2(y_2^{\mathrm{rlx}}) = E_{v_2}[\max\{0, J_2(v_2)\}]$. The optimal offer in period one is given by $J_1(y_1^{\mathrm{rlx}}(v_1, d_1 = 1)) = \zeta_{1,1}$. Hence we have

$$y_1^{\mathrm{rlx}}(v_1, d_1 = 1) = J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}]).$$

If $d = 2$, no buyer reaches his deadline in the first period, therefore the mechanism waits for period two. In period two, buyer one wins if $J_1(v_1) \geq J_2(v_2) = \zeta_{1,2}(v_2)$ and the price he has to pay in this case is given by

$$y_1^{\mathrm{rlx}}((v_1, d_1 = 2), v_2) = J_1^{-1}(\zeta_{1,2}(v_2)) = J_1^{-1}(\max\{0, J_2(v_2)\}).$$

Now suppose that the seller tries to implement the relaxed solution when the deadline is private information. Then, he has to rely on buyer one's claim about the deadline when deciding whether to make the take-it-or-leave-it-offer or to wait for

[20] A sufficient condition is positive affiliation of valuation and deadline.

the second period. To understand the dynamic pricing effect, suppose that buyer one has the highest possible valuation ($v_1 = \bar{v}$) and deadline two. The decision to reveal the deadline truthfully only depends on the expected payments, since the buyer will get the object regardless of his report. Expected payments are

$$y_1^{\text{rlx}}(\bar{v}, d_1 = 1) = J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}]) \quad \text{if he reports } d = 1,$$

$$\text{and} \qquad E_{v_2}[y_1^{\text{rlx}}((\bar{v}, d_1 = 2), v_2)] = E_{v_2}[J_1^{-1}(\max\{0, J_2(v_2)\})] \quad \text{if he reports } d = 2.$$

Obviously the expected payment is strictly smaller for $d = 1$ if J_1^{-1} is convex (J_1 is concave). It is greater (equal) if J_1 is convex (linear). Therefore, this type of buyer one will not reveal his deadline truthfully, if J_1 is concave.

If the seller is interested in value-maximization rather than revenue, valuations are not transformed by the virtual valuation function. The take-it-or-leave-it-offer in period one would be $E[v_2]$, and the price for buyer one in the second period would be v_2. Hence, expected payments would be the same for both deadlines and the buyer would not have an incentive to lie about his deadline. This shows that dynamic arrivals alone do not lead to the dynamic pricing effect.

If the seller maximizes revenue, but no new information arrives in the second period, the dynamic pricing effect also vanishes. To see this, suppose that the valuation of buyer one is already known in the first period. In this case the the mechanism in period two is unchanged, but the take-it-or-leave-it-offer in the first period is now given by $J_1^{-1}(\max\{0, J_2(v_2)\})$ rather than $J_1^{-1}(E_{v_2}[\max\{0, J_2(v_2)\}])$. Therefore, expected payments would be the same for both deadlines. Formally, the *strict* second order stochastic dominance in Lemma 2 depends on new arrivals. Without new arrivals, we have $\zeta_{1,1} = \zeta_{1,2}$ and the expected payment is independent of the deadline unless correlations lead to a static pricing effect.

Strict concavity of the virtual valuation is equivalent to

$$\frac{1 - F(v)}{(f(v))^2}(f(v)f''(v) - 2(f'(v))^2) < f'(v).$$

This implies that all distributions with an increasing density that is not too convex have strictly concave virtual valuations. Conversely, decreasing densities that are not too concave imply weak convexity of the virtual valuation.

Table 1 shows densities and virtual valuations for several distributions. For the first group, the virtual valuation is strictly concave wherever it is non-negative. For the second group, it is linear and for the third group it is convex. If valuation and deadline of a buyer are independently distributed, the relaxed solution violates incentive compatibility for all distributions in the first group and satisfies incentive

density (support: $[0,1]$)	$J(v)$	$J''(v)$
$2v$	$\frac{1}{2}\frac{3v^2-1}{v}$	$-\frac{1}{v^3} < 0$
$1-k+2kv$ ($k \in (0,1]$)	$\frac{2v-2kv+3kv^2-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} < 0$
$(k+1)v^k$ ($k > 0$)	$\frac{vk+2v-v^{-k}}{k+1}$	$-v^{-2-k}k < 0$
$12(v-\frac{1}{2})^2$	$\frac{2}{3}\frac{v^2(4v-3)}{(2v-1)^2}$	$-\frac{4}{(2v-1)^4} < 0$
$\frac{3}{2}-6(v-\frac{1}{2})^2$	$\frac{8v^2-v-1}{6v}$	$-\frac{1}{3v^3} < 0$
$2-2v$	$\frac{3v}{2}-\frac{1}{2}$	0
1 (uniform)	$2v-1$	0
$(1+k)(1-v)^k$	$\frac{(k+2)v-1}{k+1}$	0
$1-k+2kv$ ($k \in [-1,0)$)	$\frac{2v-2kv+3kv-1}{1-k+2kv}$	$-\frac{2k(1+k)^2}{(1-k+2kv)^3} > 0$

TABLE 1. Distributions with strictly concave, linear, and strictly convex virtual valuations.

compatibility for all other examples. An example for a violation of incentive compatibility for the dependent case is $f_1(v|1) = 2-2v$ and $f_1(v|2) = 1$. In this case, the virtual valuation is linear for both distributions but strictly decreasing in the deadline. If we exchange $f_1(v|1)$ and $f_1(v|2)$, incentive compatibility is satisfied for buyers with types $(1, v, 2)$. Other examples are easily constructed.

Remark: Lemma 2 conditions on the state in the arrival period. This implies that the incentive compatibility result of Theorem 2 also holds if buyers can condition their reports on the state at their arrival time. In other words, under the conditions of part (ii) of the theorem, the relaxed solution is periodic ex-post incentive compatible. This shows that the optimal solution does not rely on the seller's ability to conceal information from earlier periods.

6. The General Solution

In cases where the relaxed solution is not incentive compatible, the analysis is significantly more complex. For tractability, we restrict the model to the case of two periods ($T = 2$) and one object ($K = 1$), and assume deterministic arrival of one buyer in each period ($\nu_{1,1} = \nu_{2,1} = 1$). Furthermore, we will make an assumption that ensures that the optimal mechanism does not use lotteries in the first period (Assumption 2 below). For this case, we solve $\mathcal{R}$ subject to (M), (ICD$^{\text{d}}$) and (PE).

In the following section, we will simplify the notation and decompose the seller's problem into two sub-problems: one for $d_1 = 1$ and one for $d_1 = 2$. These problems are only linked by the incentive compatibility constraint for the deadline (ICD$^{\text{d}}$). In Section 6.2, we show that Assumption 2 rules out lotteries and solve the revenue maximization problem for $d_1 = 1$. Section 6.3 deals with the problem for $d_1 = 2$ in the regular case where the monotonicity constraint is slack. Assumption 1 guarantees that in the optimal solution, (M) is slack for buyer two. For buyer one, however, it is not sufficient for monotonicity. In Section 6.4, we show how the mechanism has to be ironed if (M) is binding at the optimal solution. The reader may want to skip section 6.4 at the first read. Finally, we combine the solutions for $d_1 = 1$ and $d_1 = 2$ to a solution of the general problem.

6.1. Decomposition of the seller's problem

Since $N_1 = N_2 = 1$, we write d, ρ, $f_2(v_2)$, and $F_2(v_2)$ instead of d_1, $\rho_{1,1}$, $f_2(v_2|2)$, and $F_2(v_2|2)$, respectively. Winning probabilities are written

$$\begin{aligned} x_1(v_1, 1) &= x_1\left(\, \xi_1 = (1) \,|\, s_1 = ((1, v_1, 1), 1)\,\right), \\ x_1(v_1, 2, v_2) &= x_1\left(\, \xi_2 = (1, 0) \,|\, s_2 = (((1, v_1, 2), (2, v_2, 2)), 1)\,\right), \\ \text{and} \qquad x_2(v_1, d, v_2) &= x_2\left(\, \xi_2 = (0, 1) \,|\, s_2 = (((1, v_1, d), (2, v_2, 2)), 1)\,\right). \end{aligned}$$

$x_1(v_1, 1)$ is the probability that buyer one gets the object if his deadline is one. $x_i(v_1, d, v_2)$ is the probability that buyer i gets the object in period two, conditional on the event that the object has not been allocated in the first period. Note that x is feasible if and only if for all $v_1, v_2 \in [0, \overline{v}]$, $d \in \{1, 2\}$, and $i \in \{1, 2\}$,

$$x_1(v_1, 1), x_i(v_1, d, v_2) \in [0, 1] \quad \text{and} \quad x_1(v_1, 2, v_2) + x_2(v_1, 2, v_2) \leq 1. \tag{F}$$

The feasibility constraint for $d = 1$ is fulfilled automatically because $x_2(v_1, 1, v_2)$ is the winning probability of buyer two conditional on the event that the object has not been allocated in the first period.

Interim winning probabilities of buyer one are given by

$$q_1(v_1, 1) = x_1(v_1, 1), \qquad \text{and} \qquad q_1(v_1, 2) = \int_0^{\overline{v}} x_1(v_1, 2, v_2) f_2(v_2) dv_2.$$

The interim winning probability of buyer two, conditional on the deadline of buyer one and the event that the object has not been allocated in period one, is given by

$$q_2(v_2, d) = \int_0^{\overline{v}} x_2(v_1, d, v_2) f_1(v_1|d) dv_1.$$

Hence, we have

$$q_2(v_2) = \rho \left(\int_0^{\overline{v}} (1 - x_1(v_1, 1)) f_1(v_1|1) d_1 \right) q_2(v_2, 1) + (1 - \rho) q_2(v_2, 2).$$

With these definitions, $\mathcal{R}$ subject to (ICD$^{\mathrm{d}}$), (PE), and (M) for buyer one, can be rewritten as the maximization problem $\mathcal{P}$:

$$\max_q \rho \int_0^{\overline{v}} \left[q_1(v_1,1)J_1(v_1|1) + (1-q_1(v_1,1)) \int_0^{\overline{v}} q_2(v_2,1)J_2(v_2)f_2(v_2)dv_2 \right] f_1(v_1|1)dv_1$$
$$+ (1-\rho) \int_0^{\overline{v}} q_1(v,2)\, J_1(v|2)\, f_1(v|2) + q_2(v,2)\, J_2(v)\, f_2(v)\, dv \qquad (\mathcal{P})$$

such that q is the reduced form of a feasible allocation rule and subject to

$$\forall d \in \{1,2\}, \forall v, v' \in [0,\overline{v}]: \qquad v > v' \Rightarrow q_1(v,d) \geq q_1(v',d), \qquad (\mathrm{M}_1)$$
$$\forall d \in \{1,2\}, \forall v \in [0,\overline{v}]: \quad U_1(v,d) = \int_0^v q_1(s,d)ds, \qquad (\mathrm{PE}_1)$$
$$\forall v \in [0,\overline{v}]: \quad U_1(v,1) \leq U_1(v,2), \text{ with equality if } v=0. \quad (\mathrm{ICD}^{\mathrm{d}}_1)$$

Except for the incentive constraint for the deadline (ICD$^{\mathrm{d}}_1$), the expected revenues for $d=1$ (first line in the objective) and $d=2$ (second line) can be maximized independently. In order to decompose the seller's problem, we introduce a function $U:[0,\overline{v}] \to [0,\overline{v}]$, $U(0)=0$, that separates $U_1(.,1)$ from $U_1(.,2)$:

$$\forall v \in [0,\overline{v}]: \qquad U_1(v,1) \leq U(v) \leq U_2(v,2), \text{ with equality if } v=0. \qquad (\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}})$$

Using U as a parameter, the maximization problem can be rewritten as $\mathcal{P}'$:

$$\max_U \rho\, \pi_1[U] + (1-\rho)\, \pi_2[U] \qquad (\mathcal{P}')$$

$\pi_1[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is one and the expected payoff of the first buyer is constrained by $U_1(v,1) \leq U(v)$ for all $v \in [0,\overline{v}]$. This maximization problem is called $\mathcal{P}_1$:

$$\pi_1[U] := \max_{q_i(.,1)} \int_0^{\overline{v}} \Bigg[q_1(v_1,1)J_1(v_1|1) + \qquad (\mathcal{P}_1)$$
$$(1-q_1(v_1,1)) \int_0^{\overline{v}} q_2(v_2,1)J_2(v_2)f_2(v_2)dv_2 \Bigg] f_1(v_1|1)dv_1$$
$$\text{s.t.} \quad q_i(v,1) \in [0,1],\ (\mathrm{PE}_1),\ (\mathrm{M}_1) \text{ and } (\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}})$$

$\pi_2[U]$ is defined as the maximal expected revenue that can be achieved if the deadline is two and the utility of the first buyer is constrained by $U_1(v,2) \geq U(v)$ for all $v \in [0,\overline{v}]$. This maximization problem is called $\mathcal{P}_2$:

$$\pi_2(U) := \max_{q_i(.,2)} \int_0^{\overline{v}} q_1(v,2)J_1(v|2)f_1(v|2) + q_2(v,2)J_2(v)f_2(v)dv \qquad (\mathcal{P}_2)$$
$$\text{s.t.} \quad (\mathrm{F}),(\mathrm{PE}_1),\ (\mathrm{M}_1) \text{ and } (\mathrm{ICD}^{\mathrm{d}}_{\mathrm{U}}).$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are solved for the same U, we get a solution for $\mathcal{P}$. The following lemma shows that $\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$ has to be checked only for the highest valuation if the seller does not use lotteries in the first period.

Lemma 3. *If $x_1(v_1, 1) \in \{0, 1\}$ for all $v_1 \in [0, \overline{v}]$, then ($ICD_U^d$) holds for any v, if it is fulfilled for $v = 0$ and $v = \overline{v}$.*

Proof. $q_1(v, 1)$ jumps from zero to one at $v = \overline{v} - U_1(\overline{v}, 1)$ if the allocation is deterministic. Therefore, the utility schedule for $d = 1$ is the lowest schedule that is consistent with $U_1(0, 1)$, $U_1(\overline{v}, 1)$ and (PE). If $U_1(0, 1) = U_1(0, 2)$ and $U_1(\overline{v}, 1) \leq U_1(\overline{v}, 2)$, then $U_1(v, 2)$ must necessarily be greater or equal than $U_1(v, 1)$ for all $v \in [0, \overline{v}]$. □

This result is very useful. It implies that the points where the constraint is binding is independent of the solution, as long as the seller does not use lotteries in the first period. In particular, since $U_1(0, 1) = U_2(0, 2) = 0$ the incentive constraint for the deadline is reduced to a single inequality.

6.2. Solution to $\mathcal{P}_1$

If ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$) is ignored, $\mathcal{P}_1$ is equivalent to the problem of finding the optimal selling strategy for a sequence of short-lived buyers. The optimal solution is a sequence of fixed prices (Riley and Zeckhauser, 1983). Optimal prices are determined working backwards in time. If the object was not sold in the first period, the optimal price in the second period is $r_2 = v_2^0$. Hence, the option value of postponing the allocation is $V_2^{\mathrm{opt}} := \int_{v_2^0}^{\overline{v}} J_2(v_2) f_2(v_2) dv_2 = v_2^0(1 - F_2(v_2^0))$. Consequently, the optimal price in the first period, r_1, is given by $J_1(r_1|1) = V_2^{\mathrm{opt}}$. This is the relaxed solution of $\mathcal{P}_1$.

If constraint ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$) is imposed, the optimal solution to $\mathcal{P}_1$ may involve lotteries.[21] To rule out this possibility we make

Assumption 2. $J_1(v|1) f_1(v|1)$ *is strictly increasing for all* $v \in [v_1^0|1, \overline{v}]$.

Lemma 3 implies that if the allocation rule is deterministic in the first period, ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$) reduces to $U_1(\overline{v}, 1) \leq \bar{U}$, where we define $\bar{U} := U(\overline{v})$. We will thus treat π_1 as a function of $\bar{U}$ and write $\pi_1(\bar{U})$ instead of $\pi_1[U]$ in this case. The optimal fixed price in period one is now given by the lowest price that satisfies $J_1(r_1|1) \geq V_2^{\mathrm{opt}}$ and $\overline{v} - r_1 \leq \bar{U}$. The optimal price in period two is not affected by constraint ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$).

[21]The no-haggling result of Riley and Zeckhauser (1983) is a consequence of a special structure of the feasible set of the maximization problem. Manelli and Vincent (2007) show that the set of extremal points of the feasible set, which contains the maximizers, is equal to the set of deterministic allocation rules. Due to the additional constraint ($\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}}$), the set of extremal points changes. Rather than trying to extend the results of Manelli and Vincent here, we use Assumption 2 as a sufficient condition for a deterministic mechanism.

Theorem 3. *Suppose f_1 satisfies Assumption 2. Then,*

(i) the optimal solution of $\mathcal{P}_1$ does not use lotteries. It is given by

$$q_1(v_1, 1) = \begin{cases} 0, & \text{if } J_1(v_1|1) < \max\{V_2^{\text{opt}}, J_1(\overline{v} - \bar{U}|1)\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$q_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ 1, & \text{otherwise}. \end{cases}$$

(ii) $\pi_1(\bar{U})$ is continuously differentiable for $\bar{U} \in (0, \overline{v})$ and strictly concave in $\bar{U}$ for $\bar{U} < \overline{v} - J_1^{-1}(V_2^{\text{opt}})$.

Proof. The proof can be found in the supplementary appendix □

To understand the role of Assumption 2, note that in the constraint $U(v) \geq \int_0^v q_1(s, 1)ds$, winning probabilities are not weighted in the integral because incentive compatibility constraints are independent of the buyer's own distribution function. In the objective, however, $q_1(v_1, 1)$ is weighted by $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$. Increasing the winning probability $q_1(v_1, 1)$ for valuations in $[v, v + \varepsilon]$, and decreasing it by the same amount on $[v', v' + \varepsilon]$, with $v' + \varepsilon \leq v$, decreases $U_1(v_1, 1)$ for $v_1 \in [v', v + \varepsilon]$ and leaves $U_1(v_1, 1)$ unchanged otherwise. Hence, such a change in q_1 does not destroy incentive compatibility. On the other hand, this shift of winning probability from low to high types increases the seller's revenue if $(J_1(v_1) - V_2^{\text{opt}})f_1(v_1)$ is increasing. Assumption 2 guarantees that $(J_1(v_1|1) - V_2^{\text{opt}})f_1(v_1|1)$ is increasing whenever $J_1(v_1|1) - V_2^{\text{opt}} \geq 0$. Therefore, the winning probability must jump from zero to one at some point and the allocation is deterministic.

If Assumption 2 does not hold, raising the winning probability for a lower valuation may be more profitable than for a higher valuation because it is sufficiently more likely that buyer one has the low valuation. For this to be the case, the decrease in the density must outweigh the increase in expected revenue, i.e. the virtual valuation. Finally, note that Assumption 2 is a sufficient condition. Presumably, a necessary and sufficient condition cannot be stated as a local condition.

6.3. Solution to $\mathcal{P}_2$ – The Regular Case

In this section, we solve $\mathcal{P}_2$, imposing ($\text{ICD}_\text{U}^\text{d}$) only for $v = \overline{v}$. By Lemma 3 and Theorem 3, this is sufficient for the general problem if Assumption 2 is fulfilled. In the derivation of the optimal solution of $\mathcal{P}_2$, however, Assumption 2 is not used. Therefore, the results of this and the following section also apply if the mechanism designer is exogenously restricted to set a fixed price in the first period.

To state the optimal solution, we define the *generalized virtual valuation* of buyer one:

$$J_1^{p_U}(v) := J_1(v|1) + \frac{p_U}{f_1(v|1)}.$$

The parameter p_U determines the magnitude of the distortion of the allocation rule away from Myerson's (1981) solution for $\mathcal{P}_2$ without $(\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}})$. ($p_U$ is the multiplier of the constraint $(\mathrm{ICD}_{\mathrm{U}}^{\mathrm{d}})$ in the underlying control problem.) Suppose we already know the optimal p_U. Then, the optimal allocation rule is given by

$$\begin{aligned} x_1(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_1^{p_U}(v_1) < \max\{0, J_2(v_2)\} \\ 1, & \text{otherwise,} \end{cases} \\ x_2(v_1, 2, v_2) &= \begin{cases} 0, & \text{if } J_2(v_2) \le \max\{0, J_1^{p_U}(v_1)\} \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{6.1}$$

For every $\bar{U} \in [0, \overline{v})$, let $p^*_{\bar{U}}$ be the lowest value $p_U \ge 0$, such that the reduced form of (6.1) satisfies $\int_0^{\overline{v}} q_1(v, 2)dv \ge \bar{U}$.

Theorem 4. *Fix $\bar{U}$ and suppose $J_1^{p^*_{\bar{U}}}(v_1)$ is strictly increasing in v_1. Then*
(i) the reduced form of (6.1) for $p_U = p^*_{\bar{U}}$ is an optimal solution of $\mathcal{P}_2$ subject to (M$_1$), (PE$_1$), and (ICD_U^d) for $v = \overline{v}$.
(ii) $p^*_{\bar{U}} = -\pi_2'(\bar{U})$.
(iii) π_2 is weakly concave.

Proof. Theorem 4 is a special case of Theorem 5 below. □

If the relaxed solution is incentive compatible, p_U is zero and valuations (v_1, v_2) tie if $J_1(v_1|2) = J_2(v_2)$, as in Myerson's solution. If the relaxed solution is not incentive compatible, p_U is strictly positive and valuations tie if $J_1^{p_U}(v_1) = J_2(v_2)$, which is equivalent to

$$(J_1(v_1|2) - J_2(v_2))f_1(v_1|2) = -p_U. \tag{6.2}$$

Figure 6.1 sketches both cases for identically distributed valuations ($f_1(.|2) = f_2$). The solid line is the Myerson-line, at which valuations tie in the relaxed solution. The dashed line is the distorted Myerson-line, at which valuations tie in the general solution. Note that for $p_U > 0$, valuations tie in an area where the (standard) virtual valuation of buyer one is strictly smaller than the virtual valuation of buyer two.

To understand condition (6.2), consider the effect on π_2 of an increase of $q_1(., 2)$. Fix any (v_1, v_2) on the distorted Myerson-line, such that $0 \le J_1^{p_U}(v_1) \le \overline{v}$. In the figure, this corresponds to $\alpha \le v_1 \le \beta$. In order to increase $q_1(v_1, 2)$, the allocation has to be changed from buyer two to buyer one at (v_1, v_2). This leads to a marginal

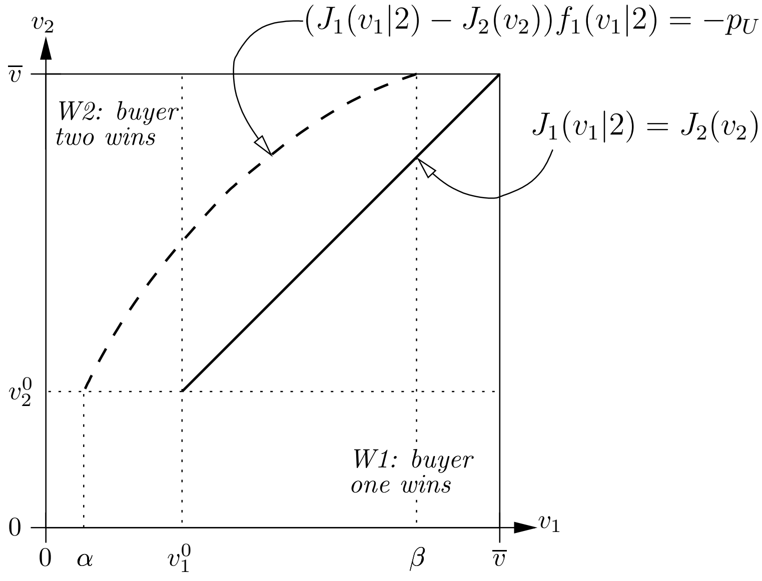

FIGURE 6.1. Optimal allocation rule

change in π_2 of $J_1(v_1|2) - J_2(v_2) < 0$ per mass of type profiles for which the allocation is changed. This mass of type profiles is proportional to $f_1(v_1|2)$. Hence, the left-hand side of (6.2) quantifies the marginal cost of increasing $q_1(v_1, 2)$.

The marginal cost of increasing $q_1(v_1, 2)$ must be independent of v_1. The reason is that winning probabilities are not weighted in the constraint $\int_0^{\overline{v}} q_1(s, 2)ds \geq \bar{U}$. If the marginal cost of changing $q_1(v_1, 2)$ varied with v_1, we could increase $q_1(v_1, 2)$ where the marginal cost is small and decrease it where the marginal cost is big. If we chose this variation such that $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s, 2)ds$ were not changed, we could increase the objective function without violating the constraints—a contradiction. Hence, the marginal cost of increasing $q_1(., 2)$ must be constant and equal to p_U for all $v_1 \in [\alpha, \beta]$. As the utility of the highest type is given by $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1(s, 2)ds$, p_U can also be interpreted as the marginal cost of the constraint $U_1(\overline{v}, 2) \geq \bar{U}$.

Furthermore, note that the distortion is increasing in p_U, and that by Assumption 1, the marginal cost of a distortion is increasing in the distance from the Myerson solution (the LHS of (6.2) is decreasing in v_2). Therefore, (a) it is optimal to choose the lowest p_U such that ($\text{ICD}_\text{U}^\text{d}$) is satisfied, and (b) the cost of distortions is convex, which implies concavity of π_2 in $\bar{U}$.

Finally, (6.2) implies that the distortion of the Myerson-line is bigger for types with lower densities. Intuitively, the expected cost of a distortion is lower for types that are less frequent. This also implies that an increasing density can lead to non-monotonicities of the winning-probability.

6.4. Solution to $\mathcal{P}_2$ – The Irregular Case

To ensure an increasing winning probability for buyer one, Theorem 4 requires that $J_1^{p_U^*}$ is strictly increasing. This is a condition on an endogenous object and Assumption 1 does not guarantee monotonicity of $J_1^{p_U}$ for all values of p_U. A decreasing density $f_1(v|2)$ together with Assumption 1 would be sufficient, but this is quite restrictive and rules out most of the examples of concave virtual valuations in Table 1. To give a complete solution without further assumptions, we show that Myerson's ironing procedure can be used to deal with non-monotonicities of $J_1^{p_U}$.

Definition 4 (Ironing; Myerson, 1981)**.** (i) For every $t \in [0,1]$, define

$$M_1^{p_U}(t) := J_1(F_1^{-1}(t|2)|2) + \frac{p_U}{f_1(F_1^{-1}(t|2)|2)},$$

as the generalized virtual valuation at the t-quantile of $F_1(.|2)$.

(ii) Integrate this function:

$$H^{p_U}(t) := \int_0^t M_1^{p_U}(s)ds.$$

(iii) Take the convex hull (i.e. the greatest convex function G such that $G(t) \leq H^{p_U}(t)$ for all t):

$$\bar{H}^{p_U}(t) := \operatorname{conv} H^{p_U}(t).$$

(iv) Since $\bar{H}^{p_U}$ is convex, it is almost everywhere differentiable and any selection $\bar{M}_1^{p_U}(t)$ from the sub-gradient is non-decreasing.

(v) Reverse the change of variables made in (i) to obtain the *ironed generalized virtual valuation*

$$\bar{J}_1^{p_U}(v_1) := \bar{M}_1^{p_U}(F_1(v_1|2)).$$

In the irregular case, the optimal allocation rule depends on two parameters, p_U and $\underline{x}_1^0$, and has the following structure:

$$\begin{aligned} \bar{x}_1(v_1,2,v_2) &= \begin{cases} 1, & \text{if } \bar{J}_1^{p_U}(v_1) > 0 \text{ and } \bar{J}_1^{p_U}(v_1) \geq J_2(v_2) \\ \underline{x}_1^0, & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \text{ and } J_2(v_2) \leq 0, \\ 0, & \text{otherwise,} \end{cases} \quad , \\ \bar{x}_2(v_1,2,v_2) &= \begin{cases} 0, & \text{if } J_2(v_2) \leq \max\{0, \bar{J}_1^{p_U}(v_1)\}, \\ 1, & \text{otherwise.} \end{cases} \end{aligned} \tag{6.3}$$

The parameters are determined as follows. First, let p_U^* be the minimal $p_U \geq 0$ such that the reduced form of (6.3) with $\underline{x}_1^0 = 1$ satisfies $\int_0^{\overline{v}} q_1(v,2)dv \geq \bar{U}$. Second, if $p_U^* > 0$, select $\underline{x}_1^{0*} \in [0,1]$ such that $\int_0^{\overline{v}} q_1(v,2)dv = \bar{U}$, otherwise set $\underline{x}_1^{0*} = 1$.

The additional parameter $\underline{x}_1^0$ is only needed if $\bar{J}_1^{p_U}(v_1) = 0$ on an interval $[\underline{v}_1^0, \bar{v}_1^0]$ with $\underline{v}_1^0 < \bar{v}_1^0$. In this case, $\int_{\underline{v}_1^0}^{\bar{v}_1^0} J_1^{p_U}(v)dv = 0$ and hence, $U_1(\overline{v}, 2)$ can be varied at constant marginal cost p_U by changing the winning probability for all valuations in the interval $[\underline{v}_1^0, \bar{v}_1^0]$. Therefore, a single value of p_U defines the ironed generalized virtual valuation for different values $\bar{U}$ in a non-empty interval $[a, b]$. $\underline{x}_1^0$ is varied to achieve different values of $U_1(\overline{v}, 2) \in [a, b]$.

The allocation rule in (6.3) excludes buyer one if his valuation is smaller than $\underline{v}_1^0$. With a valuation in $[\underline{v}_1^0, \bar{v}_1^0]$, he can win against buyer two if $v_2 \leq v_2^0$, but he gets the object only with probability $\underline{x}_1^0$.[22] To summarize, we have

Theorem 5. *(i) The reduced form of (6.3) for p_U^* and $\underline{x}_1^{0*}$ is an optimal solution of $\mathcal{P}_2$ subject to (M_1), (PE_1), and (ICD_U^d) for $v = \overline{v}$.*
(ii) For almost every $\bar{U}$, $\pi_2'(\bar{U}) = -p_U^*$.
(iii) π_2 is weakly concave in $\bar{U}$ and strictly concave if $p_U > 0$ and $\bar{J}_1^{p_U}(v) = 0$ has a unique solution.

Proof. See Appendix B. □

Note that if $J_1^{p_U}$ is increasing, $\bar{J}_1^{p_U}$ equals $J_1^{p_U}$. Therefore, Theorem 4 is a special case of Theorem 5.

6.5. Global Solution and Discussion

Under Assumption 2, $\mathcal{P}'$ reduces to the problem of choosing $\bar{U}$ optimally. The first order necessary condition is

$$\rho\, \pi_1'(\bar{U}) = -(1 - \rho)\, \pi_2'(\bar{U}).$$

By Theorem 5, π_2 is concave and by Theorem 3 and Assumption 2, π_1 is concave. Therefore, the first-order condition is also sufficient. To determine the optimal distortion, it suffices to compute the unique solution $(\bar{U}, p_U)$, $p_U \geq 0$ of

$$p_U = \frac{\rho}{1 - \rho}\, \pi_1'(\bar{U}),$$

$$\text{and} \qquad \bar{U} \leq \int_0^{\overline{v}} q_1^{p_U}(v, 2)dv_1, \quad \text{with equality if } p_U > 0,$$

where q^{p_U} is the reduced form of (6.1) for given value of p_U.[23] An explicit form of the solution is not available. However, for given p_U, $U_1(\overline{v}, 2) = \int_0^{\overline{v}} q_1^{p_U}(v, 2)dv_1$ is easy to

[22] It is also possible to construct a deterministic allocation rule with the same reduced form. Choose $\hat{v}_2$ such that $\underline{x}_1^0 = \frac{F_2(\hat{v}_2)}{F_2(v_2^0)}$. For $v_1 \in [\underline{v}_1^0, \bar{v}_1^0]$, set $x_1(v_1, 2, v_2) = 1$ if $v_2 \leq \hat{v}_2$ and $x_1(v_1, 2, v_2) = 0$ otherwise. This construction has the disadvantage, however, that the allocation decision for buyer one depends on truthful reports of buyer two in cases when he can never win the object.

[23] We only discuss the global solution for the regular case. The irregular case is similar.

calculate and an explicit expression for π_1' is given in the proof of Theorem 3. Hence, it is easy to compute the optimal p_U numerically. If Assumption 2 is violated, π_1 may fail to be concave and it may be necessary to compute all local maxima to find the global solution. We will now discuss several properties of the general solution.

Monotonicity of q_2. $q_2(v_2, 1)$, defined by the fixed price r_2, and $q_2(v_2, 2)$, defined by the reduced form of (6.3), are non-decreasing. This follows from Assumption 1. Therefore, $q_2(v_2)$ is also non-decreasing and the optimal solutions of $\mathcal{P}_1$ and $\mathcal{P}_2$ together fulfill all constraints of $\mathcal{P}$. We have derived an optimal solution of $\mathcal{P}$.

Distortions in Both Periods. By Theorem 3, $\pi_1(\bar{U})$ is continuously differentiable. Therefore, $p_U > 0$ implies that the allocation for $d = 1$ is distorted. Hence, the general solution involves a distortion for both deadlines, whenever the relaxed solution is not incentive compatible. As distortions are more costly at the deadline which occurs more frequently. The relative magnitude of the distortions depends on ρ. If $d = 1$ is relatively unlikely (ρ small), then the distortion of the fixed price is bigger and the auction is closer to Myerson's solution.

Distortions. In the first period, the fixed price is $\max\{\bar{U} - \overline{v}, J_1^{-1}(V_2^{\text{opt}}|1)\}$. It is distorted upwards compared to the relaxed solution, to make the fixed price less attractive. To analyze the distortions in the auction in period two, note that

$$\forall v_1 \in [0, \overline{v}] : \quad J_1^{p_U}(v_1) = J_1(v_1|2) + \frac{p_U}{f_1(v_1|2)} > J_1(v_1|2),$$

if the relaxed solution is not incentive compatible ($p_U > 0$). Therefore, the reserve price for buyer one is smaller than in the relaxed solution. Secondly, for all valuations above the reserve price, the winning probability is higher than in the relaxed solution because v_1 ties with a higher valuation v_2. Finally, in contrast to the relaxed solution, the winning probability of bidder two is strictly smaller than one for all $v_2 \in [0, \overline{v}]$. For every $p_U > 0$, there is a non-empty interval $(c, \overline{v}]$ such that $J_1^{p_U}(v_1) > \overline{v}$ for all valuations $v_1 \in (c, \overline{v}]$. Buyer two cannot win against buyer one if $v_1 > c$.

Bunching. We find that for the optimal allocation rule, there is bunching at the top of the type space if the incentive compatibility constraint for the deadline is binding. The bunching region has full dimension. The optimal mechanism does not separate different types of buyer one if their valuations are very high ($v_1 > c$). In the auction, these types win with probability one and have to make an expected payment equal to the fixed price in the first period. Therefore, we have bunching of valuations as well as deadlines. This finding is robust: a (small) bunch occurs even if the allocation is only slightly distorted.

Dominant Strategies and Indirect Implementation. There are several ways to implement the optimal auction in period two. For example, it can be implemented by a generalized Vickrey auction. In this auction, the winning bidder pays the valuation for which his (generalized) virtual valuation ties with the (generalized) virtual valuation of the losing bidder. For buyer two, this mechanism is incentive compatible in dominant strategies.[24] Hence, the optimal mechanism does not rely on the seller's ability to conceal information about period one.

As in the static auction model, there is also an open format that corresponds to this direct mechanism. Consider the following ascending clock auction. The auctioneer has a clock that runs from zero to $\overline{v}$. For each bidder i, the auctioneer's clock value c_a is translated into a bidder-specific clock value c_i. For bidder one, this is $c_1 = \left(J_1^{pU}\right)^{-1}(c_a|2)$. For bidder two, this is $c_2 = J_2^{-1}(c_a)$. The auctioneer raises c_a continuously and bidders can drop out at any time. If bidder i drops out, the clock stops immediately. Bidder $j \neq i$ wins the object and has to make a payment equal to his bidder-specific clock-value c_j. Given the informational assumptions made in this paper, this auction is strategically equivalent to the generalized Vickrey auction. It has the advantage that the winning bidder does not have to reveal his true valuation to the auctioneer.

7. Conclusion

We have analyzed a dynamic mechanism design model, in which a seller wants to maximize the revenue from selling one or multiple units of a good to buyers that arrive over time, within a finite time horizon. The main innovation of the model is that buyers are privately informed about their deadlines for buying the good.

We found sufficient conditions for full separation. In this case, the incentive compatibility constraint for the deadline is slack in the seller-optimal mechanism. The relaxed solution, which neglects the constraint is fully optimal. We also found sufficient conditions for violations of the neglected constraints. Both conditions exploit (a) a static pricing effect that depends on stochastic dependencies between the deadline and the valuation of a buyer, and (b) a dynamic pricing effect that depends on non-linearities in the virtual valuation function of a buyer. While the former effect can also be found in static models with two-dimensional private information, the latter effect is due to the dynamic nature of the allocation problem. The critical virtual valuation that buyer has to overbid in order to get a unit is a martingale with respect to the information about all buyer's types. Therefore, critical virtual

[24] If the auction is considered in isolation, it is also a dominant strategy for buyer one to bid his true valuation. In the dynamic context, however, it is not a dominant strategy to report the deadline truthfully.

valuations for later deadlines are mean preserving spreads of critical virtual valuations for earlier deadlines. This leads to lower (higher) payoffs for later deadlines in the case of concave (convex) virtual valuations and destroys (guarantees) incentive compatibility.

We have also studied the case of bunching. If the relaxed solution is not incentive compatible, the incentive constraint for the deadline is binding in the optimal mechanism. Therefore, we have to solve a mechanism design problem with two-dimensional private information. The fact that the second dimension is a deadline puts some structure on the model. The two-dimensional problem is similar to a standard one-dimensional mechanism design problem with a type-dependent outside option. We solve this model for the case of two time periods, one object and deterministic arrival of one buyer in each period. We show that the optimal mechanism has a very similar structure as the relaxed solution, but the allocation rule is distorted in favor of buyers with later deadlines and earlier arrival. This provides incentives to report the deadline truthfully. The optimal mechanism can be described in terms of a generalized virtual valuation.

Several assumptions were made to ensure tractability or to simplify the exposition.

Discounting. Throughout the paper, we have abstracted from discounting. This assumption can be relaxed. If only payments are discounted and buyers and the seller use a common discount factor, the analysis is almost identical. On the other hand, if valuations are discounted, Lemma 1 may not be valid. For example, it may be optimal to allocate a unit in the first period even if the deadline of the winner is two, because the waiting cost due to discounting is too high. In this case, it is more complicated to rule out upward deviations in the deadline.

The appropriate modeling choice depends on the application. In the example given in the introduction, the buyer's valuation is the present discounted value of the revenue stream from the contractual relationship with the third party. This could for example be a production contract. If production starts after the deadline and is independent of the time at which the firm obtains the object (as long as it gets it before the deadline), it seems reasonable that the firm only discounts payments. Similar arguments apply in any situation where the buyer plans to use the object at a fixed time after the deadline as in the case of flight tickets of hotel reservations.

Stochastic Exit. We have implicitly assumed that buyers are available until their deadline. In some situations, however, buyers may find other opportunities to purchase a similar object if the seller does not sell in the period of arrival. Therefore, stochastic exit, random participation as in Rochet and Stole (2002) or competition with other sellers would be interesting extensions for future research.

Incentive Compatibility of the Relaxed Solution with Many Objects. For more than two time periods, the proof of the martingale property of the critical virtual valuation uses a property of the optimal allocation rule that is shown in Mierendorff (2009b) for the case of a single object. With one object, there is a unique bidder in each period that has a positive probability of winning. This greatly simplifies the analysis because in each state, the type of only one buyer is relevant for the allocation rule and buyers who are irrelevant in period t will not be recalled in the future. Unfortunately, a generalization of this property to the case of many objects is not available. I conjecture, however, that the martingale property of the critical virtual valuation generalizes to the case of many objects. If this conjecture is true, then the dynamic as well as the static pricing effect, and the absence of a competition effect will carry over to the case of multiple objects and more than two time periods. Therefore, the sufficient conditions for incentive compatibility of the relaxed solution will also apply to the more general model.

Privately Known Arrival Times. The arrival time has similar properties as the deadline. Misreports are only feasible in one direction and the arrival time does not enter the utility function directly. Therefore, the analysis of a model with private arrival times is similar to the analysis in the present paper. In particular there is a static pricing effect. Pai and Vohra (2008b) show that the relaxed solution is incentive compatible with respect to the arrival time if virtual valuations are decreasing in the arrival time. The dynamic pricing effect, however, does not arise with arrival times. The arrival time does not influence the time of the allocation, and therefore the amount of information available to the seller is independent of the arrival time. We also note that there is an additional effect that relaxes the incentive compatibility constraint for the arrival time. By delaying the report of his arrival, a buyer runs the risk that units are allocated to buyers that he could have overbid if he had reported his arrival truthfully. Therefore, an adverse static pricing effect does not automatically destroy incentive compatibility.

Generalizing the Bunching Case: More Bidders. Introducing more bidders who arrive in the second period is straight forward. The assumption that there is only one bidder in the first period is more important. It was used to show that the object is offered to buyer one for a fixed price if he reports deadline one. We have shown that in this case, misreporting deadline one instead of deadline two is most profitable for the buyer with the highest valuation. Hence, we know exactly where the incentive compatibility constraint for the deadline binds. If more than one buyer arrives in the first period, a fixed price is no longer optimal and the incentive compatibility constraint for the deadline may bind for interior types. The exact points where it binds arise endogenously in the optimal solution.

Generalizing the Bunching Case: Number of Periods. Increasing the number of periods introduces several complications. Consider for example a model with three periods. Suppose that in each period a single bidder arrives, whose deadline can be any period after his arrival. Now, from period two onwards, there is more than one bidder who participates in the mechanism. This introduces similar problems as the introduction of more bidders in the first period, as discussed in the preceding paragraph. Additional complications will arise because buyers from different periods will have to be treated asymmetrically. In the third period, the mechanism designer has to design an optimal auction with three different bidders, two of which have type-dependent participation constraints. In the case of two periods and two bidders, the feasibility constraint could be used to eliminate the winning probability of one bidder (see Appendix B). A generalization of this approach to three bidders is not obvious.

Appendix A. Proofs of Lemma 2 and Theorem 2

Proof of Lemma 2. The result will be shown separately for $K = 1$ and $T = 2$.

Case 1 ($K = 1$)*:* To simplify notation, define $c_a^\tau := \max_{j\in\{i\in I_a|d_i=\tau\}} J_a(v_j|\tau)$ and $c_{\leq a}^\tau := \max\{c_1^\tau, \dots, c_a^\tau\}$. For fixed $i \in I_{\leq\tau}$ define $c_a^{\tau,-i} := \max_{j\in\{l\in I_a\setminus\{i\}|d_l=\tau\}} J_a(v_j|\tau)$ and $c_{\leq a}^{\tau,-i} := \max\{c_1^{\tau,-i}, \dots, c_a^{\tau,-i}\}$.

Results from Mierendorff (2009b) imply that for each state s_t, in which the object is still available, there is a unique period $\theta_t \geq t$, in which the object will be allocated if it is allocated to a buyer $i \in I_{\leq t}$. θ_t is determined by

$$\begin{aligned} & c_{\leq t}^\tau \leq E_{s_{\tau+1}}\left[V_{\tau+1}(s_{\tau+1})|s_\tau = s_t, k_{\tau+1} = 1\right] \qquad \forall \tau < \theta_t, \\ \text{and} \qquad & c_{\leq t}^{\theta_t} > E_{s_{\theta_t+1}}\left[V_{\theta_t+1}(s_{\theta_t+1})|s_{\theta_t} = s_t, k_{\theta_t+1} = 1\right]. \end{aligned}$$

Furthermore, there is a unique tentative winner $i_t^* \in I_{\leq t}$ in state s_t. i_t^* has deadline $d_{i_t^*} = \theta_t$, and virtual valuation $J_{a_{i_t^*}}(v_{i_t^*}|d_{i_t^*}) = c_{\leq t}^{\theta_t}$. For all other buyers in $I_{\leq t}$, the winning probability conditional on s_t is zero. Hence, in order to compute the value function in state s_t, H_t can be replaced $(\theta_t, c_{\leq t}^{\theta_t})$:

$$\begin{aligned} V_T(s_T) &= \mathbf{1}_{\{k_T=1\}} \max\{0, J_{a_{i_T^*}}(v_{i_t^*}|T)\} \\ &= \mathbf{1}_{\{k_T=1\}} \max\{0, c_{\leq T}^T\} =: V_T((\theta_T, c_{\leq T}^T)), \end{aligned}$$

$$\text{and} \qquad V_t(s_t) = \begin{cases} J_{a_{i_t^*}}(v_{i_t^*}|t), & \text{if } d_{i_t^*} = t \text{ and } k_t = 1, \\ E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|s_t\right], & \text{otherwise.} \end{cases}$$
$$= \begin{cases} c_{\leq t}^{\theta_t} & \text{if } \theta_i = t \text{ and } k_t = 1, \\ E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|(\theta_t, c_{\leq t}^{\theta_t}), k_{t+1} = 1\right], & \text{otherwise.} \end{cases}$$
$$=: V_t((\theta_t, c_{\leq t}^{\theta_t})),$$

In order to compute the critical virtual valuation of the winning buyer, of course, more information is needed. Suppose buyer i arrives in period a_i and $k_{a_i} = 1$. Then, he wins in period d_i, if and only if

$$\forall t \in \{a_i, \ldots, d_i\} \qquad c_{\leq t}^t \leq V_t((d_i, J_{a_i}(v_i|d_i)), k_t = 1),$$
$$\text{and} \qquad J_{a_i}(v_i|d_i) > E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})|s_{d_i}, k_{d_i+1} = 1\right],$$

where we define the expected value in the last line as zero if $d_i = T$. To give an expression for the critical virtual valuation, we define

$$z_t^d(c_{\leq t}^t) = \min\left\{z \geq 0 \middle| c_{\leq t}^t = E_{s_{t+1}}\left[V_{t+1}(s_{t+1})|(d,z), k_{t+1} = 1\right]\right\}$$

where here and in the following, $E_{s_{t+1}}[\ldots|(d,z), k_{t+1} = 1] = E_{s_{t+1}}[\ldots|(\theta_t, c_{\leq\theta_t}^{\theta_t}) = (d,z), k_{t+1} = 1]$. With this definition we have

$$\zeta_{a_i,d_i}^i(H_{d_i}, 1) = \max\Big\{z_{a_i}^{d_i}(c_{\leq a_i}^{a_i,-i}), \ldots, z_{d_i-1}^{d_i}(c_{\leq d_i-1}^{d_i-1,-i}), c_{\leq d_i}^{d_i,-i},$$
$$E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})|H_{d_i,-i}, k_{d_i+1} = 1\right]\Big\}.$$

Claim 1. $z_t^d(c_{\leq t}^t) = z_{d-1}^d(z_t^{d-1}(c_{\leq t}^t))$.

Proof of Claim 1. If $z_t^d(c_{\leq t}^t) = 0$ then also $z_t^{d-1}(c_{\leq t}^t) = 0$ and therefore $z_{d-1}^d(z_t^{d-1}(c_{\leq t}^t)) = 0$. Suppose $z_t^d(c_{\leq t}^t) > 0$ and $z_t^{d-1}(c_{\leq t}^t) > 0$. This implies

$$c_{\leq t}^t = E_{s_{t+1}}\left[V_{t+1}(s_{t+1})\middle|(d-1, z_t^{d-1}(c_{\leq t}^t)), k_{t+1} = 1\right]$$
$$= E_{s_{t+1}}\left[V_{t+1}(s_{t+1})\middle|(d, z_t^d(c_{\leq t}^t)), k_{t+1} = 1\right]$$

The second equation is equivalent to

$$E_{s_{t+1}}\Big[\max\Big\{c_{t+1}^{t+1}, E_{s_{t+2}}\big[\ldots \tag{A.1}$$
$$E_{s_{d-1}}\left[\mathbf{1}_{\{k_{d-1}=1\}}\max\left\{c_{\leq d-1}^{d-1}, z_t^{d-1}(c_{\leq t}^t), E_{s_d}\left[V_s(s_d)|H_{d-1}, k_d = 1\right]\right\}\middle|s_{d-2}\right]$$
$$\ldots\,\big|s_{t+1}\big]\Big\}\Big|I_{\leq t} = \emptyset, k_{t+1} = 1\Big]$$
$$= E_{s_{t+1}}\Big[\max\Big\{c_{t+1}^{t+1}, E_{s_{t+2}}\big[\ldots$$
$$E_{s-1}\left[\mathbf{1}_{\{k_{d-1}=1\}}\max\left\{c_{\leq d-1}^{d-1}, E_{s_d}\left[V_s(s_d)\middle|(d, z_t^d(c_{\leq t}^t)), k_d = 1\right],\right.\right.$$

$$E_{s_d}\left[V_s(s_d)|H_{d-1}, k_d = 1]\right\} \left|s_{d-2}\right] \qquad \ldots \left|s_{t+1}\right]\right\} \left|I_{\le t} = \emptyset, k_{t+1} = 1\right]$$

Now suppose by contradiction that $z_t^d(c_{\le t}^t) > z_{d-1}^d(z_t^{d-1}(c_{\le t}^t))$. This implies

$$E_{s_d}\left[V_d(s_d)\middle|(d, z_t^d(c_{\le t}^t)), k_d = 1\right] > z_t^{d-1}(c_{\le t}^t).$$

Conditional on $I_{\le t} = \emptyset$, with positive probability the realization of s_{d-1} is such that $k_{d-1} = 1$ and

$$\begin{aligned} E_{s_d}\left[V_d(s_d)\middle|(d, z_t^d(c_{\le t}^t)), k_d = 1\right] &> \max\left\{c_{\le d-1}^{d-1}, E_{s_d}\left[V_d(s_d)|H_{d-1}, k_d = 1\right]\right\} \\ &> z_t^{d-1}(c_{\le t}^t). \end{aligned}$$

But this contradicts (A.1). Similarly, $z_t^d(c_{\le t}^t) < z_{d-1}^d(z_t^{d-1}(c_{\le t}^t))$ leads to a contradiction. This proves the claim. $\square$

Now, consider the critical virtual valuation for deadline $d_i - 1$:

$$\begin{aligned} \zeta_{a_i,d_i-1}^i(H_{d_i-1}, 1) = \max\Big\{& z_{a_i}^{d_i-1}(c_{\le a_i}^{a_i,-i}), \ldots, z_{d_i-2}^{d_i-1}(c_{\le d_i-2}^{d_i-2,-i}), c_{\le d_i-1}^{d_i-1,-i}, \\ & E_{s_{d_i}}\left[V_{d_i}(s_{d_i})|H_{d_i-1,-i}, k_{d_i} = 1\right]\Big\}. \end{aligned}$$

Claim 1 allows us to replace the cutoff values $z_\tau^{d_i-1}$. $\forall \tau \in \{a_i, \ldots, d_i - 2\}$:

$$\begin{aligned} x_\tau^{d_i-1}(c_{\le\tau}^{\tau,-i}) &= E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\middle|(d_i, z_{d_i-1}^{d_i}(z_\tau^{d_i-1}(c_{\le\tau}^{\tau,-i}))), k_{d_i} = 1\right], \\ &= E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\middle|(d_i, z_\tau^{d_i}(c_{\le\tau}^{\tau,-i})), k_{d_i} = 1\right]. \end{aligned}$$

Hence,

$$\begin{aligned} &\zeta_{a_i,d_i-1}^i(H_{d_i-1}, 1) = \\ =&\max\Big\{E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\middle|(d_i, z_{a_i}^{d_i}(c_{\le a_i}^{a_i,-i})), k_{d_i} = 1\right], \ldots \\ &\quad \ldots, E_{s_{d_i}}\left[V_{d_i}(s_{d_i})\middle|(d_i, z_{d_i-1}^{d_i}(c_{\le d_i-1}^{d_i-1,-i})), k_{d_i} = 1\right], E_{s_{d_i}}\left[V_{d_i}(s_{d_i})|H_{d_i-1,-i}, k_{d_i} = 1\right]\Big\} \\ =&E_{s_{d_i}}\Big[\max\Big\{z_{a_i}^{d_i}(c_{\le a_i}^{a_i,-i}), \ldots, z_{d_i-1}^{d_i}(c_{\le d_i-1}^{d_i-1,-i}), \\ &\qquad c_{\le d_i}^{d_i}, E_{s_{d_i+1}}\left[V_{d_i+1}(s_{d_i+1})|H_{d_i,-i}, k_{d_i+1} = 1\right]\Big\}\Big|H_{d_i-1}\Big] \\ =&E_{H_{d_i}}[\zeta_{a_i,d_i}^i(H_{d_i}, 1)|H_{d_i-1}]. \end{aligned}$$

As $\zeta_{a_i,d_i-1}^i(H_{d_i-1}, 1)|H_{d_i-1}$ is deterministic for each H_{d_i-1},

$$\zeta_{a_i,d_i-1}^i(H_{d_i-1}, 1)|H_{d_i-1} \succ_{\text{SSD}} \zeta_{a_i,d_i}^i(H_{d_i}, 1)|H_{d_i-1}$$

and the lemma follows.

Case 2 ($T = 2$)*:* Now we revert to the notation from the main text and use $c_{(K)}^t$. Let $c_{(k)}^{t,-i}$ denote the k[th] highest virtual valuation among the buyers with deadline t

in $I_{\leq t}\backslash\{i\}$. Fix any state $s_1 = (H_1, K)$. Let K_1 denote the number of units that are allocated in period one in state $(H_{1,-i}, K-1)$. We distinguish two sub-cases.

Case A—In state $(H_{1,-i}, K)$, K_1 units are allocated in the first period: If, in state $((H_{1,-i}, (1, v_i, 1)), K)$, buyer i gets a unit in the first period, then the remaining $K-1$ units are allocated as in state $(H_{1,-i}, K-1)$. This means that K_1 units are allocated to buyers other than i in period one and $K - K_1 - 1$ units are retained. Hence, i's virtual valuation must exceed the option value of retaining the $K - K_1^{\text{st}}$ unit. We have

$$\begin{aligned}\zeta^i_{1,1}(H_1, K) &= E_{s_2}\left[V_2(s_2)|H_{1,-i}, k_2 = K - K_1\right] - E_{s_2}\left[V_2(s_2)|H_{1,-i}, k_2 = K - K_1 - 1\right], \\ &= E_{s_2}\left[\max\left\{0, c^{2,-i}_{(K-K_1)}\right\}\Big|H_1\right].\end{aligned}$$

In state $((H_{1,-i}, (1, v_i, 2)), K)$, the number of units that are allocated in the first period must also be K_1. It is obvious that the arrival of buyer i with $d_i = 2$ cannot increase the number of units allocated in the first period. On the other hand, suppose that in state $((H_{1,-i}, (1, v_i, 2)), K)$, only $K_1 - 1$ units are allocated in the first period. Then

$$\begin{aligned}c^{1,-i}_{(K_1)} &\leq E_{s_2}\left[\max\left\{0, c^2_{(K-K_1+1)}\right\}\big|(H_{1,-i}, (1, v_i, 2))\right], \\ &\leq E_{s_2}\left[\max\left\{0, c^2_{(K-K_1+1)}\right\}\big|(H_{1,-i}, (1, \overline{v}, 2))\right], \\ &= E_{s_2}\left[\max\left\{0, c^2_{(K-K_1)}\right\}\big|H_{1,-i}\right], \\ &< c^{1,-i}_{(K_1)},\end{aligned}$$

where the last inequality follows from our assumption that in state $(H_{1,-i}, K-1)$, K_1 units are allocated in the first period. This is a contradiction. But if K_1 objects are allocated in the first period, then

$$\zeta^i_{1,2}(H_2, K) = \max\left\{0, c^{2,-i}_{(K-K_1)}\right\}.$$

Hence, in case A, $E_{s_2}[\zeta^i_{1,2}(H_2, K)|H_1] = \zeta^i_{1,1}(H_1, K)$ and $\zeta^i_{1,1}(H_1, K)|H_1 \succ_{\text{SSD}} \zeta^i_{1,2}(H_2, K)|H_1$.

Case B—In state $(H_{1,-i}, K)$, K_1+1 objects are allocated in the first period: Again, if in state $((H_{1,-i}, (1, v_i, 1)), K)$, buyer i gets an object in the first period, then the remaining $K-1$ objects are allocated as in state $(H_{1,-i}, K-1)$. Hence, in case B we have

$$\zeta^i_{1,1}(H_1, K) = c^{1,-i}_{(K_1+1)}.$$

In state $((H_{1,-i},(1,v_i,2)),K)$, it depends on v_i, how many objects are retained for the second period. Define z by

$$c^{1,-i}_{(K_1+1)} = E_{s_2}\left[\max\left\{0, c^2_{(K-K_1)}\right\}\big|(H_{1,-i},(1,J_1^{-1}(z|2),2))\right].$$

If $J_1(v_i|2) \geq z$, then $K-K_1$ objects are retained, otherwise only $K-K_1-1$ objects are retained. Hence, we have

$$\zeta^i_{1,2}(H_2,K) = \begin{cases} c^{2,-i}_{(K-K_1)}, & \text{if } z < c^{2,-i}_{(K-K_1)}, \\ z & \text{if } c^{2,-i}_{(K-K_1)} \leq z < c^{2,-i}_{(K-K_1-1)}, \\ c^{2,-i}_{(K-K_1-1)} & \text{if } c^{2,-i}_{(K-K_1-1)} \leq z. \end{cases}$$

Note that for $H_1 = (H_{1,-i},(1,J_1^{-1}(z|2),2))$ this equals $\max\left\{0, c^2_{(K-K_1)}\right\}$. Therefore, also in case B, $E_{s_2}[\zeta^i_{1,2}(H_2,K)|H_1] = \zeta^i_{1,1}(H_1,K)$ and $\zeta^i_{1,1}(H_1,K)|H_1 \succ_{\text{SSD}} \zeta^i_{1,2}(H_2,K)|H_1$. □

Proof of Theorem 2. Consider a buyer i with type (a,v,d), where $a < d \leq T$ and let $d' \in \{1,\ldots,d-1\}$. Fix the state in the arrival period s_a, and let

$$\begin{aligned} G(\zeta) &= \text{Prob}\left\{\zeta^i_{a,d}(H_d,k_a) \leq \zeta\big|s_a\right\}, \\ \text{and} \qquad G'(\zeta) &= \text{Prob}\left\{\zeta^i_{a,d'}(H_{d'},k_a) \leq \zeta\big|s_a\right\}. \end{aligned}$$

Lemma 2 implies that G and G' have the same mean and $G' \succ_{\text{SSD}} G$.

(i) Suppose $v = \overline{v}$, $J_a(v|d')$ is strictly concave and $J_a(v|d') \geq J_a(v|d)$ for all $v \in [v^0_a|d', \overline{v}]$. Conditional on s_a the expected payoff of i is given by

$$\begin{aligned} U_a(\overline{v},d) &= \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d))dG(\zeta), \\ &\leq \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d'))dG(\zeta), \\ &< \int_0^{\overline{v}} (\overline{v} - J_a^{-1}(\zeta|d')dG'(\zeta) = U_a(\overline{v},d'). \end{aligned}$$

In the second line we have used that $J_a^{-1}(\zeta|d') \leq J_a^{-1}(\zeta|d)$ for $\zeta > 0$. In the third line we have used strict convexity of $J_a^{-1}(\zeta|d')$ as a function of ζ. A similar argument can be made if $J_a(v|d)$ is strictly concave. If $J_a(v|d') > J_a(v|d)$ for all $v < \overline{v}$, the first inequality becomes strict and strict concavity can be replaced by weak concavity.

(ii) Suppose $J_a(v|d')$ is convex and $J_a(v|d) \geq J_a(v|d')$ for all $v \in [v^0_a|d, \overline{v}]$. Conditional on s_a we have

$$U_a(v,d) = \int_0^{J_a(v|d)} (v - J_a^{-1}(\zeta|d))dG(\zeta),$$

$$\geq \int_0^{J_a(v|d')} (v - J_a^{-1}(\zeta|d'))dG(\zeta),$$
$$= J_a^{-1}(0|d')G(0) + \int_0^{J_a(v|d')} \frac{d}{d\zeta} J_a^{-1}(\zeta|d')\, G(\zeta)d\zeta,$$
$$\geq J_a^{-1}(0|d')G'(0) + \int_0^{J_a(v|d')} \frac{d}{d\zeta} J_a^{-1}(\zeta|d')\, G'(\zeta)d\zeta = U_a(v,d')$$

The last line follows because $\frac{d}{d\zeta}J_a^{-1}(v|d')$ is non-negative and non-increasing and for all non-negative and non-increasing functions $\phi : [0,\overline{v}] \to [0,\overline{v}]$, we have

$$\forall x \in [0,\overline{v}] : \qquad \int_0^x \phi(s)G'(s)ds \leq \int_0^x \phi(s)G(s)ds.$$

For $\phi(s) = \mathbf{1}_{\{s \leq x\}}$ this follows directly from SSD and since any non-increasing function $\phi : [0,\overline{v}] \to [0,\overline{v}]$ can be uniformly approximated by non-increasing step functions the result follows. A similar argument can be made if $J_a(v|d)$ is convex. □

Appendix B. Proof of Theorem 5

It will be convenient to make the changes of variables $t_1 = F_1(v_1|2)$ and $t_2 = F_2(v_2)$. Defining $v_1(t_1) := F_1^{-1}(t_1|2)$ and $v_2(t_2 := F_2^{-1}(t_2)$, we have

$$\begin{aligned} t_i &\sim U[0,1] \quad \text{for } i = 1,2, \\ v_1'(t_1) &= \frac{1}{f_1(v_1(t_1)|2)}, \\ \text{and} \quad v_2'(t_2) &= \frac{1}{f_2(v_2(t_2))}, \end{aligned}$$

Furthermore, for $i = 1,2$ we introduce

$$\begin{aligned} q_i(t) &= q_i(v_i(t),2), \\ U(t) &= U_1(v_1(t),2), \\ M_1(t) &= J_1(v_1(t)|2) = v_1(t) - (1-t)v_1'(t) \\ M_2(t) &= J_2(v_2(t)) = v_2(t) - (1-t)v_2'(t) \\ t_1^0 &= F_1(v_1^0|2\,|2). \\ \text{and} \qquad t_2^0 &= F_2(v_2^0). \end{aligned}$$

The objective of the seller becomes

$$R[q_1,q_2] := \int_0^1 q_1(t)M_1(t) + q_2(t)M_2(t)dt. \tag{B.1}$$

The following Theorem formulates the feasibility constraint in terms of q.

Theorem 6 (Mierendorff, 2009a). *For $i = 1, 2$, let $q_i : [0,1] \to [0,1]$ be non-decreasing. (q_1, q_2) is the reduced form of a feasible allocation rule if and only if for all $t_1, t_2 \in [0,1]$,*

$$\int_{t_1}^{1} q_1(t)dt + \int_{t_2}^{1} q_2(t)dt \leq 1 - t_1 t_2.$$

Now we can restate $\mathcal{P}_2$ as $\mathcal{P}_2'$:

$$\pi_2(\bar{U}) = \sup_{(q_1,q_2)} R[q_1, q_2] \qquad (\mathcal{P}_2')$$

subject to

$$\forall t \in [0,1]: \qquad q_i(t) \in [0,1], \qquad \text{(B.2)}$$

$$\forall t > t', \qquad q_i(t) \geq q_i(t'), \qquad \text{(B.3)}$$

$$\forall t_1, t_2 \in [0,1]: \qquad \int_{t_1}^{1} q_1(\theta)d\theta + \int_{t_2}^{1} q_2(\theta)d\theta \leq 1 - t_1 t_2, \qquad \text{(B.4)}$$

$$\forall t \in [0,1]: \qquad U(t) = \int_0^t q_1(\theta)v_1'(\theta)d\theta, \qquad \text{(B.5)}$$

$$\text{and} \qquad U(1) \geq \bar{U}. \qquad \text{(B.6)}$$

Using $q_i(F_i(v_i|2)) = q_i(v_i, 2)$, a solution to $\mathcal{P}_2$ can be derived easily from a solution to $\mathcal{P}_2'$.

We can use the (non-standard) constraint (B.4) to eliminate q_2 from the objective function. For $q_1 : [0,1] \to [0,1]$ non-decreasing, define the *inverse* as

$$q_1^{-1}(t) := \begin{cases} 1 & \text{if } q_1(1) < t, \\ \inf\{\theta \in [0,1] \mid q_1(\theta) \geq t\} & \text{otherwise.} \end{cases}$$

Lemma 4. *Let $q_1 : [0,1] \to [0,1]$ be non-decreasing. Then an optimal solution to*

$$\sup_{q_2} \int_0^1 q_2(t)M_2(t)dt \qquad \textit{subject to (B.2)–(B.4),}$$

is given by

$$q_2^*(t) = \begin{cases} q_1^{-1}(t) & \textit{if } t \geq t_2^0, \\ 0 & \textit{otherwise.} \end{cases}$$

The solution is unique for almost every t.

Proof. The proof can be found in the supplementary appendix. □

Using Lemma 4, (B.1) becomes

$$\int_0^1 q_1(t)M_1(t)dt + \int_{t_2^0}^1 q_1^{-1}(t)M_2(t)dt. \qquad \text{(B.7)}$$

If q_1 is absolutely continuous, substituting $s = q_1(t)$ in the second integral yields

$$\int_0^1 q_1(t)M_1(t) + tq_1'(t)\tilde{M}_2(q_1(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt, \tag{B.8}$$

where we define $\tilde{M}_2(t) := \max\{0, M_2(t)\}$.

Monotonicity implies some regularity of q_1. In particular $q_1 = q_1^C + q_1^J$ where q_1^C is a continuous function and q_1^J is a pure jump function. This leaves two problems unresolved. Firstly, we have to deal with jumps and secondly, absolute continuity of q_1^C is not guaranteed. To deal with this we restrict q_1 to be globally Lipschitz continuous with constant K,

$$q_1 \in \mathcal{L}^K := \{q : [0,1] \to [0,1] \mid \forall t, t' \in [0,1] : \; |q(t) - q(t')| \leq K|t - t'|\}.$$

We define the maximization problem $\mathcal{P}_2^K$ as $\mathcal{P}_2'$ subject to the additional constraint $q_1 \in \mathcal{L}^K$. The set of winning probabilities that satisfy (B.4) is weakly-compact (cf. Mierendorff (2009a) and Border (1991)). Since $\mathcal{L}^K$ is sequentially compact standard arguments can be used to prove existence.

Theorem 7. *(a) An optimal solution of $\mathcal{P}_2'$ exists.*
(b) For every $K > 0$, an optimal solution of $\mathcal{P}_2^K$ exists.

Proof. The proof can be found in the supplementary appendix. □

The next step is to show that Lipschitz solutions converge to the general solution if K tends to infinity. The proof is based on Reid (1968).

Lemma 5. *Let $(q_1^n, q_2^n)_{n\in\mathbb{N}}$ a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $K_n \to \infty$ as $n \to \infty$. Then, there exists a solution (q_1, q_2) of $\mathcal{P}_2'$ and a sub-sequence $(q_1^{n_j}, q_2^{n_j})_{j\in\mathbb{N}}$ such that $q_i^{n_j}(t) \overset{j\to\infty}{\longrightarrow} q_i(t)$ for almost every t and $R[q_1, q_2] = \pi_2(\bar{U})$.*

Proof. After taking a sub-sequence, we can assume that (q_1^n, q_2^n) converges a.e. to a solution $(\hat{q}_1, \hat{q}_2)$ of $\mathcal{P}_2'$ (see proof of Theorem 7). To show optimality of $(\hat{q}_1, \hat{q}_2)$, let (q_1, q_2) be an optimal solution of $\mathcal{P}_2'$. We can extend q_1 to $\mathbb{R}$ by setting $q_1(t) = 0$ if $t < 0$ and $q_1(t) = 1$ if $t > 1$. Define $q_{d,1} : \mathbb{R} \to [0,1]$ as

$$q_{d,1}(t) := \frac{1}{2d} \int_{t-d}^{t+d} q_1(s)ds.$$

By the Lebesgue differentiation theorem $q_{d,1}(t) \to q_1(t)$ for almost every $t \in [0,1]$ as $d \to 0$. Since q_1 is non-decreasing and $q_1(t) \in [0,1]$, $q_{d,1}$ also has these properties. Furthermore $q_{d,1} \in \mathcal{L}^{\frac{1}{2d}}$:

$$\forall t > t' : \qquad 0 \leq q_{d,1}(t) - q_{d,1}(t') \;=\; \frac{1}{2d}\left(\int_{t-d}^{t+d} q_1(s)ds - \int_{t'-d}^{t'+d} q_1(s)ds\right)$$

$$\begin{aligned}
&= \frac{1}{2d}\left(\int_{t'+d}^{t+d} q_1(s)ds - \int_{t'-d}^{t-d} q_1(s)ds\right) \\
&\leq \frac{1}{2d}\int_{t'+d}^{t+d} q_1(s)ds \\
&\leq \frac{1}{2d}(t-t')
\end{aligned}$$

Since $q_{d,1}$ may violate $\int_0^1 q_{d,1}(t)v_1'(t)dt \geq \bar{U}$, we define $\tilde{q}_{d,1} := \lambda_d + (1-\lambda_d)q_{d,1}$ and

$$\tilde{q}_{d,2}(t) := \begin{cases} \tilde{q}_{d,1}^{-1}(t), & \text{if } M_2(t) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda_d := \max\left\{0, \frac{\bar{U}-\int_0^1 q_{d,1}(t)v_1'(t)dt}{\bar{v}-\int_0^1 q_{d,1}(t)v_1'(t)dt}\right\}$. For every d, $(\tilde{q}_{d,1}, \tilde{q}_{d,2})$ is a solution of $\mathcal{P}_2^{\frac{1}{2d}}$. λ_d converges to zero as $d \to 0$. By Lemma 4, $q_2(t) = q_1^{-1}(t)$ for a.e. t such that $M_2(t) \geq 0$ and $q_2(t) = 0$ otherwise. Hence, for $i = 1, 2$, $\tilde{q}_{d,i} \to q_i$ almost everywhere as $d \to 0$. By the dominated convergence theorem, $R[\tilde{q}_{d,1}, \tilde{q}_{d,2}] \to R[q_1, q_2]$ and $R[q_1^n, q_2^n] \to R[\hat{q}_1, \hat{q}_2]$. Define $d_n = \frac{1}{2K_n}$. Then, $R[\tilde{q}_{d_n,1}, \tilde{q}_{d_n,2}] \leq R[q_1^n, q_2^n]$ and we have $R[q_1^n, q_2^n] \to R[q_1, q_2]$ and hence $R[\hat{q}_1, \hat{q}_2] = R[q_1, q_2]$. □

In the next section, we derive properties of the Lipschitz solution. Finally, we show that there is a limiting solution that yields the same expected revenue as the solution proposed in Theorem 5.

B.1. Solution on the class $\mathcal{L}^K$

Using Lemma 4, we rewrite $\mathcal{P}_2^K$ as a control problem. The state variables are the expected utility of bidder one, denoted $U(t)$, and the winning probability, denoted $q(t)$, (in the control problem we write q instead of q_1). As q is absolutely continuous, we can use $u(t) = q'(t)$ as a control variable. The objective is defined as

$$R_c[U, q, u] := \int_0^1 q(t)M_1(t) + tu(t)\tilde{M}_2(q(t))dt + \int_{q(1)}^1 \tilde{M}_2(t)dt.$$

where u is a measurable control

$$u : [0, 1] \to [0, K]. \tag{B.9}$$

The evolution of the state variables is governed by

$$U'(t) = q(t)v_1'(t), \tag{B.10}$$

$$q'(t) = u(t). \tag{B.11}$$

We impose the state constraint

$$\forall t \in [0,1]: \qquad q(t) \leq 1. \tag{B.12}$$

Furthermore, we impose the following constraints on the start- and endpoints:

$$U(0) = 0, \tag{B.13}$$

$$q(0) \geq 0, \tag{B.14}$$

$$(1) \geq \bar{U}, \tag{B.15}$$

To summarize, we have the following control problem:

$$\max_{(U,q,u)} R_c[U,q,u], \qquad \text{subject to (B.9)–(B.15).} \tag{$\mathcal{P}_C^K$}$$

(B.10) is (B.5) in differential form. (B.9) and (B.11) ensure that $q \in \mathcal{L}^K$ and non-decreasing. (B.9), (B.11) and (B.14) imply $q(t) \geq 0$ for all t. Hence, we can dispense with a second state constraint.

The Pontriyagin maximum principle yields the following necessary conditions for an optimum.

Theorem 8 (Clarke (1983), pp. 210-212)**.** *Let (U,q,u) be a solution of $\mathcal{P}_C^K$. If (U,q,u) is optimal, there exists $\omega \in \{0,1\}$, an absolutely continuous function $p : [0,1] \to \mathbb{R}^2$, the components of which we denote by (p_U, p_q), and a non-negative measure μ on $[0,1]$, such that the following conditions hold:*

(i) For almost every $t \in [0,1]$,

$$p'_U(t) = 0, \tag{B.16}$$

$$p'_q(t) = -\omega\left[M_1(t) + tu(t)\tilde{M}'_2(q(t))\right] - p_U v'_1(t). \tag{B.17}$$

(ii) For almost every $t \in [0,1]$, $u(t)$ maximizes

$$\left[\omega t \tilde{M}_2(q(t)) + p_q(t) + \mu[0,t)\right] u.$$

(iii) μ is supported on $\{q(t) = 1\}$,

(iv) p satisfies the transversality conditions

$$p_q(0) \leq 0, \qquad \text{(with equality if } q(0) > 0,)$$

$$p_U(1) \geq 0, \qquad \text{(with equality if } U(1) > \underline{U},)$$

$$p_q(1) = -\omega\tilde{M}_2(q(1)) - \mu[0,1].$$

(v) $\omega + \|p\| + \|\mu\| > 0$.

Note that (B.16) implies that p_U is constant. First, we show that trivial solutions do not occur.

Lemma 6 (Non-triviality)**.** *If $\bar{U} < \overline{v}$, $\omega = 1$.*

Proof. Suppose that $\omega = 0$. By (B.17), $p_q'(t) = -p_U v_1'(t)$. By the transversality conditions, $p_U \geq 0$. $p_U = 0$ implies, $p_q'(t) = 0$ and $p_q(t) = p_q(0)$ for all t. $p_U > 0$ implies, $p_q'(t) < 0$ and $p_q(t) < 0$ for all $t > 0$.

Suppose $p_U > 0$. By, the transversality condition this implies $U(1) = \bar{U}$. By (ii), $u(t)$ maximizes $(p_q(t) + \mu[0,t))u$. If $q(0) < 1$, $\mu[0,t) = 0$ for t close to zero and hence $u(t) = 0$. As $\mu[0,t)$ cannot become positive we must have $q(t) = q(0) < 1$ for all t and consequently $\mu[0,1] = 0$. The transversality condition therefore requires $p_q(1) = 0$, a contradiction. If, however, $q(0) = 1$ we would have $U(1) = \overline{v} > \bar{U}$. Again a contradiction.

Now suppose that $p_U = 0$. If $q(1) < 1$, $\mu[0,1] = 0$ and by the transversality conditions, $p(t) = 0$ for all t. This implies $\omega + \|p\| + \|\mu\| = 0$, in contradiction to (v). Hence, $q(1) = 1$. Since $p_q(t) = p_q(1)$, we have $p_q(t) = -\mu[0,1]$. To fulfill (v) we must have $\mu[0,1] > 0$. $u(t)$ maximizes $(\mu[0,t) - \mu[0,1])u$. This implies that $u(t) = 0$ if $q(t) < 1$. Hence, we must have $q(t) = 1$ for all $t \in [0,1]$. This implies $U(1) = \overline{v}$, which cannot be optimal if $\bar{U} < \overline{v}$. $\square$

Defining $M_1^{p_U}(t) := M_1(t) + p_U v_1'(t)$, we can rewrite (B.17) as

$$-p_q'(t) = M_1^{p_U}(t) + tu(t)\tilde{M}_2'(q(t)), \text{ for a. e.} \, t \in [0,1]. \tag{B.18}$$

Condition (ii) implies that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) > 0, \tag{B.19}$$

$$u(t) \in [0,K] \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) = 0, \tag{B.20}$$

$$u(t) = 0 \qquad \text{if } t\tilde{M}_2(q(t)) + p_q(t) + \mu[0,t) < 0. \tag{B.21}$$

In (B.19), $\mu[0,t)$ was omitted because $q(t) < 1$ if $u(t) = K$. Integrating (B.18) yields for $s,t \in [0,1]$:

$$\begin{aligned} p_q(t) &= p_q(s) - \int_s^t M_1^{p_U}(\theta) + \theta u(\theta)\tilde{M}_2'(q(\theta))d\theta \\ &= p_q(s) - \int_s^t M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + s\tilde{M}_2(q(s)). \end{aligned} \tag{B.22}$$

If we substitute (B.22) in (B.19)–(B.21) and define $H^{p_U}(t) = \int_0^t M_1^{p_U}(\theta)d\theta$ and $m_q(t) = \int_0^t \tilde{M}_2(q(\theta))d\theta$, we have that for almost every $t \in [0,1]$,

$$u(t) = K \qquad \text{if } p_q(0) + m_q(t) > H^{p_U}(t), \tag{B.23}$$

$$u(t) \in [0,K] \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) = H^{p_U}(t), \tag{B.24}$$

$$u(t) = 0 \qquad \text{if } p_q(0) + m_q(t) + \mu[0,t) < H^{p_U}(t). \tag{B.25}$$

Lemma 7 (Reid (1968))**.** *Suppose $p_q(0) + m_q(t) = H^{p_U}(t)$ for $t \in \{\underline{t}, \bar{t}\}$, $\underline{t} < \bar{t}$ and $q(t) < 1$ for $t < \bar{t}$. Let $\alpha, \beta \in \mathbb{R}$ and $l(t) = \alpha + \beta t$. If $l(t) \leq H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$, then $p_q(0) + m_q(t) \geq l(t)$ for all $t \in [\underline{t}, \bar{t}]$.*

Proof. Suppose that $m_q(s)+p_q(0) < l(s)$ for some $s \in (\underline{t}, \bar{t})$. Then there exists $\varepsilon > 0$ and $\underline{t} < t_1 < t_2 < \bar{t}$ such that $m_q(t)+p_q(0) < l(t)-\varepsilon$ for $t \in (t_1, t_2)$, $m_q(t_1)+p_q(0) = l(t_1) - \varepsilon$, and $p_q(0) + m_q(t_2) = l(t_2) - \varepsilon$. This implies that $m_q'(t) = \tilde{M}_2(q(t))$ cannot be constant on (t_1, t_2). On the other hand, $m_q(t) + p_q(0) + \mu[0, t) = m_q(t) + p_q(0) < l(t) - \varepsilon < H(t)$ and hence $u(t) = 0$ for $t \in (t_1, t_2)$, which implies that $m_q'(t)$ is constant, a contradiction. □

An immediate implication of the Lemma is that $p_q(0) + m_q(t) \geq \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$, where $\bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$ denotes the convex hull of H^{p_U} restricted to $[\underline{t}, \bar{t}]$, i.e. the greatest convex function $G : [\underline{t}, \bar{t}] \to \mathbb{R}$ such that $G(t) < H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$. Furthermore, $p_q(0)+ m_q(t)$ is convex because q and $\tilde{M}_2$ are non-decreasing. This yields the following

Corollary 1. *Suppose $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [\underline{t}, \bar{t}]$, with equality at the endpoints of the interval and $q(t) < 1$ for $t < \bar{t}$. Then $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$, for all $t \in [\underline{t}, \bar{t}]$.*

If $M_1^{p_U}$ is non-decreasing on $[\underline{t}, \bar{t}]$, then $H^{p_U}(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)$. Differentiating $p_q(0) + m_q(t) = \bar{H}^{p_U}_{[\underline{t},\bar{t}]}$ yields $M_1^{p_U} = \tilde{M}_2(q(t))$ for $t \in [\underline{t}, \bar{t}]$.

If, however, $M_1^{p_U}$ is not monotonic on $[\underline{t}, \bar{t}]$, differentiating yields $\bar{M}^{p_U}_{[\underline{t},\bar{t}]}(t) = \tilde{M}_2(q(t))$, where $\bar{M}^{p_U}_{[\underline{t},\bar{t}]} = \frac{d\bar{H}^{p_U}_{[\underline{t},\bar{t}]}(t)}{dt}$ is non-decreasing. Hence, Reid's Lemma provides a control theoretic technique to show that Myerson's ironing procedure can be used to solve irregular instances of mechanism design problems.

Now we establish some properties of the optimal solution. Define

$$x_{p_U}(t) = \begin{cases} 0, & \text{if } M_1^{p_U}(t) < M_2(0), \\ M_2^{-1}(M_1^{p_U}(t)), & \text{if } M_1^{p_U}(t) \in [M_2(0), \overline{v}], \\ 1, & \text{if } M_1^{p_U}(t) > \overline{v}, \end{cases}$$

and
$$x^{[\underline{t},\bar{t}]}_{p_U}(t) = \begin{cases} 0, & \text{if } \bar{M}^{p_U}_{[\underline{t},\bar{t}]}(t) < M_2(0), \\ M_2^{-1}(\bar{M}^{p_U}_{[\underline{t},\bar{t}]}(t)), & \text{if } \bar{M}^{p_U}_{[\underline{t},\bar{t}]}(t) \in [M_2(0), \overline{v}], \\ 1, & \text{if } \bar{M}^{p_U}_{[\underline{t},\bar{t}]}(t) > \overline{v}. \end{cases}$$

The derivative of x_{p_U} is given by

$$x_{p_U}'(t) = \frac{M_1'(t) + p_U v''(t)}{M_2'(x_{p_U}(t))}.$$

The assumptions on f_i and F_i guarantee that $x'_{p_U}(t)$ is continuous on $[0,1]$. Let $K^{p_U} := \max_{t\in[0,1]} x'_{p_U}(t)$. Then $x_{p_U} \in \mathcal{L}^{K^{p_U}}$. In what follows, we write $\bar{H}^{p_U}$ for $\bar{H}^{p_U}_{[0,1]}$ and $\bar{M}^{p_U}_1$ for $\bar{M}^{p_U}_{[0,1]}$.

Lemma 8 (interior solution). *Suppose $u(t) \in (0, K)$ for a.e. $t \in [\underline{t}, \bar{t}]$, $\underline{t} < \bar{t}$. Then for all $t \in [\underline{t}, \bar{t}]$,*

(i) $q(t) = x_{p_U}(t)$ if $q(t) \geq t^0_2$,

(ii) $M^{p_U}_1(t) = 0$ if $q(t) < t^0_2$.

Proof. If $u(t) > 0$, we must have $\mu[0, t) = 0$. (B.23) – (B.25) imply that $p_q(0) + m_q(t) = H^{p_U}(t)$ for all$t \in (\underline{t}, \bar{t})$. Differentiating this w.r.t. t yields

$$\tilde{M}_2(q(t)) = M^{p_U}_1(t).$$

If $q(t) \geq t^0_2$, $\tilde{M}_2(q(t)) = M_2(q(t))$ and hence that $q(t) = x_{p_U}(t)$. If $q(t) < t^0_2$, $\tilde{M}_2(q(t)) = 0$ and hence $M^{p_U}_1(t) = 0$. By continuity, the results extend to $\underline{t}$ and $\bar{t}$. □

Next, we derive necessary conditions for intervals where $u(t)$ is in $\{0, K\}$.

Lemma 9 (constant q). *Suppose $q(t) = a \in [0, 1]$ on $[\underline{t}, \bar{t}]$, $\underline{t} < \bar{t}$, and let $[\underline{t}, \bar{t}]$ be chosen maximally. Then*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$
$$p_q(0) + m_q(t) = H^{p_U}(t),$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \bar{t}$ if $\bar{t} < 1$, and furthermore

$$M^{p_U}_1(\underline{t}) \geq \tilde{M}_2(a), \qquad \text{if } \underline{t} > 0, \tag{B.26}$$
$$\text{and} \quad M^{p_U}_1(\bar{t}) \leq \tilde{M}_2(a), \qquad \text{if } \bar{t} < 1. \tag{B.27}$$

Proof. If $q(t)$ is constant, then for almost every $t \in (\underline{t}, \bar{t})$, $u(t) = 0$ and therefore $p_q(t) + t\tilde{M}_2(q(t)) + \mu[0, t) \leq 0$ and $p_q(0) + m_q(t) + \mu[0, t) \leq H^{p_U}(t)$. As $\mu \geq 0$ and by continuity, $p_q(t) + t\tilde{M}_2(q(t)) \leq 0$ and $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for $t \in \{\underline{t}, \bar{t}\}$.

Suppose $\underline{t} > 0$ and let $S_- := \{0 < t < \underline{t} \mid u(t) > 0\}$. Since $q(t) < a$ for $t < \underline{t}$, and q is absolutely continuous, $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \nearrow \underline{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equalities in the Lemma follow for $\underline{t} > 0$. For $\bar{t} < 1$ set $S_+ := \{\bar{t} < t < 1 \mid u(t) > 0\}$. $S_+ \cap [\bar{t}, \bar{t} + \delta]$ has positive measure for every $\delta > 0$. Hence, there exists a sequence $t_n \searrow \bar{t}$ with $p_q(t_n) + t_n\tilde{M}_2(q(t_n)) \geq 0$ and $p_q(0) + m_p(t_n) \geq H^{p_U}(t_n)$ for all n. By continuity, the first two equations in the Lemma follow for $\bar{t} < 1$.

To show (B.26), note that for almost every $t \in S_-$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (B.22) yields

$$p_q(\underline{t}) = p_q(t) - \int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - \underline{t}\tilde{M}_2(q(\underline{t})) + t\tilde{M}_2(q(t)).$$

With $p_q(\underline{t}) = -\underline{t}\tilde{M}_2(q(\underline{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_t^{\underline{t}} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \geq 0,$$

for almost every $t \in S_-$. If this inequality is fulfilled, there must be a $t' \in [t, \underline{t}]$ with

$$M_1^{p_U}(t') - \tilde{M}_2(q(t')) \geq 0.$$

As $S_- \cap [\underline{t} - \delta, \underline{t}]$ has positive measure for every $\delta > 0$, t and hence t' can be chosen arbitrarily close to $\underline{t}$. By continuity this implies

$$M_1^{p_U}(\underline{t}) - \tilde{M}_2(q(\underline{t})) \geq 0.$$

To show (B.27), note that for almost every $t \in S_+$, $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$. (B.22) yields

$$p_q(t) = p_q(\bar{t}) - \int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta - t\tilde{M}_2(q(t)) + \bar{t}\tilde{M}_2(q(\bar{t})).$$

With $p_q(\bar{t}) = -\bar{t}\tilde{M}_2(q(\bar{t}))$ and $p_q(t) + t\tilde{M}_2(q(t)) \geq 0$ this implies

$$\int_{\bar{t}}^{t} M_1^{p_U}(\theta) - \tilde{M}_2(q(\theta))d\theta \leq 0,$$

for almost every $t \in S_+$. As above there exists $t' \in [\bar{t}, t]$ such that the integrand is non-positive at t'. t and t' can be chosen arbitrarily close to $\bar{t}$. Therefore, by continuity

$$M_1^{p_U}(\bar{t}) - \tilde{M}_2(q(\bar{t})) \leq 0.$$

$\square$

Lemma 9 implies that there cannot be an interval where q is constant and $q \in (0, 1)$ if x_{p_U} is strictly increasing.

Lemma 10. *Suppose $u(t) = K$ for almost every $t \in (\underline{t}, \bar{t})$, $\underline{t} < \bar{t}$. Let $(\underline{t}, \bar{t})$ be chosen maximally. Then for $t = \underline{t}$ and for $t = \bar{t}$ if $\bar{t} < 1$,*

$$p_q(t) + t\tilde{M}_2(q(t)) = 0,$$

for $t = \underline{t}$ if $\underline{t} > 0$ and for $t = \bar{t}$ if $\bar{t} < 1$

$$p_q(0) + m_q(t) = H^{p_U}(t).$$

Furthermore,

$$M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t})), \qquad \text{if } \underline{t} > 0, \tag{B.28}$$

$$\text{and} \qquad M_1^{p_U}(\bar{t}) \geq \tilde{M}_2(q(\bar{t})), \qquad \text{if } \bar{t} \in [0,1]. \tag{B.29}$$

Proof. The proof is very similar to the proof of the preceding Lemma. To show the first equality for $\underline{t} = 0$, the transversality condition can be used to obtain $p_q(0) \leq 0$. For $\bar{t} = 1$, (B.29) follows from $M_1^{p_U}(1) \geq \overline{v}$ and $\tilde{M}_2(q(t)) \leq \overline{v}$. □

Setting $q(t) = x_0(t)$ for $t \geq t_1^0$ and $q(t) = 0$ otherwise, yields the optimal solution of Myerson (1981). This is not surprising because p_U would be zero if the incentive compatibility constraint for the deadline were ignored. The following Lemma, which does not depend on the maximum principle, excludes solutions that have lower winning probabilities than the undistorted solution x_0.

Lemma 11. *For $K > K^0$, let $b \geq t_1^0$ be the unique solution to $(b - t_1^0)K = x_0(b)$. If $q(t) \leq x_0(t)$ for all $t \in [t_1^0, 1]$ and $q(t) < x_0(t)$ for some $t \in [b, 1]$, then q is not optimal.*

Proof. Suppose by contradiction that q is an optimal solution with the properties stated in the Lemma. Let $b' \in [0, b]$ be the unique solution to $q(t_1^0) + (b' - t_1^0)K = x_0(b')$. Define

$$\tilde{q}(t) = \begin{cases} q(t), & \text{if } t < t_1^0, \\ q(t_1^0) + (t - t_1^0)K, & \text{if } t \in [t_1^0, b'], \\ x_0(t), & \text{if } t > b'. \end{cases}$$

Obviously, $\tilde{q} \in \mathcal{L}^K$ and $\tilde{U}(1) \geq \bar{U}$. Since x_0 is the optimal solution absent constraints, $\tilde{q}$ yields higher revenue than q. This contradicts the optimality of q. □

Lemma 12. *If $\bar{U} < \overline{v}$, then $p_U \leq \bar{p}_U := 1 + \max_{t \in [0,1]} \frac{\overline{v} - v_1(t)}{v_1'(t)} < \infty$.*

Proof. Suppose to the contrary that, $p_U > \bar{p}_U$. Then $M_1^{p_U}(t) > \tilde{M}_2(1) = \overline{v}$ for all $t \in [0, 1]$. By Lemma 8.ii, this implies $q(\underline{t}) \geq t_2^0$ if $u(t) \in (0, K)$ on a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 8.i, this implies $q(t) = x_{p_U}(t)$, for all $t \in [\underline{t}, \bar{t}]$, but this contradicts $u(t) > 0$ if $M_1^{p_U}(t) > \overline{v}$. Hence we have $u(t) \in \{0, K\}$ for all $t \in [0, 1]$.

Suppose $u(t) = 0$ on a maximal interval $[\underline{t}, \bar{t}]$. By Lemma 9, this implies $\bar{t} = 1$. If $u(t) = K$ on a maximal interval $[\underline{t}, \bar{t}]$, Lemma 10 implies $\underline{t} = 0$. Therefore, there exists $a \in [0, 1]$ such that $u(t) = K$ for $t < a$ and $u(t) = 0$ for $t > a$. Suppose $a > 0$. Lemma 10 implies $p_q(0) = 0$ if $a > 0$. As $M_1^{p_U}(t) > \tilde{M}_2(q(t))$ for all t, we have $p_q(t) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. Hence, $u(t) = 0$ for all $t > 0$ and $a = 0$.

If $q(t) = q$ is constant, Lemma 11 implies that $q > t_2^0$. Therefore, $p_q(0) = 0$ by the transversality condition. Using (B.22), we get $p_q(1) = -\int_0^1 M_1^{p_U}(t)dt < 0$. The transversality condition and $p_U > 0$ imply $U(1) = \bar{U}$. This yields $q = \frac{\bar{U}}{\bar{v}}$. If $q < 1$, then $\mu[0,1] = 0$, and hence, $p_q(1) = -\tilde{M}_2(q(1)) > -\int_0^1 M_1^{p_U}(t)dt$ by the transversality condition. So we must have $q = 1$ and hence $\bar{U} = \overline{v}$, which is ruled out by assumption. □

Note that $|x'_{p_U}(t)| \leq \frac{M_1'(t)+p_U|v_1'(t)|}{\min_{x\in[0,1]}|M_2'(x)|}$. Defining $\overline{K} := \max_{t\in[0,1]} \frac{M_1'(t)+\bar{p}_U|v_1'(t)|}{\min_{x\in[0,1]}|M_2'(x)|}$ we have $x_{p_U} \in \mathcal{L}^{\overline{K}}$ for all $p_U \leq \bar{p}_U$.

Lemma 13. *Let $(\underline{t}, \bar{t})$ be a maximal interval such that $u(t) = K$ for all $t \in (\underline{t}, \bar{t})$ and $K > \overline{K}$. Then $q(t) < \max\{t_2^0, x_{p_U}(t)\}$ for all $t \in [\underline{t}, \bar{t})$. If $\underline{t} > 0$, then $q(\underline{t}) < t_2^0$. Furthermore $\bar{t} < 1$.*

Proof. If $q(t) \geq \max\{t_2^0, x_{p_U}(t)\}$, then $q(\bar{t}) > \max\{t_2^0, x_{p_U}(\bar{t})\}$ because $K > \overline{K}$. Hence $\tilde{M}_2(q(\bar{t})) > M_1^{p_U}(\bar{t})$, a contradiction by Lemma 10. If $\underline{t} > 0$, $q(\underline{t}) < t_2^0$ because otherwise (B.28) and $K > \overline{K}$ would imply $q(\bar{t}) \geq \max\{t_2^0, x_{p_U}(\bar{t})\}$, which is a contradiction. Finally, $\bar{t} = 1$ would imply $q(t) < x_0(t)$ for all $t \in [t_1^0, 1)$. This is also a contradiction by Lemma 11. □

Lemma 14. *For $K > K^0$, $q(1) = 1$.*

Proof. Suppose $q(1) < 1$. By Lemma 11, $q(1) > t_2^0$. By the transversality condition, $p_q(1) = -\tilde{M}_2(q(1))$. Differentiating $p_q(t) + t\tilde{M}_2(q(t))$, we get $\frac{d}{dt}(p_q(t) + t\tilde{M}_2(q(t))) = p_q'(t) + \tilde{M}_2(q(t)) + t\tilde{M}_2'(q(t))q'(t) = \tilde{M}_2(q(t)) - M_1^{p_U}(t)$. As $q(1) < x_{p_U}(1)$ we have $\frac{d}{dt}(p_q(t) + t\tilde{M}_2(q(t))) < 0$, and thus $p(t) + t\tilde{M}_2(q(t)) > 0$ for t sufficiently close to one. Hence $u(t) = K$ on a maximal interval $[\underline{t}, 1]$. As $K > K^0$, $\underline{t} > 0$ and hence $q(\underline{t}) < t_2^0$ by Lemma 13. This contradicts optimality by Lemma 11. □

Define $c := \min\{t \mid q(t) = 1\}$. By the preceding Lemma, this is well defined for $K > K^0$.

Lemma 15. *For $K > \overline{K}$,*

$$\begin{aligned} p_q(0) + m_q(c) &= H^{p_U}(c), \\ p_q(c) + c\tilde{M}_2(q(c)) &= 0, \\ M_1^{p_U}(c) &= \tilde{M}_2(1). \end{aligned}$$

Proof. If $c < 1$ the first two equations are implied by Lemma 9. If $c = 1$, $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to one. ($u(t) = 0$ is ruled out by $c = 1$, $u(t) \neq K$ follows from the same argument as in the proof of Lemma 14). Hence, the first two equalities hold for t close to c and by Lemma

8 also the third equality holds for t close to c. By continuity the equalities also hold for c. If $c < 1$, $M_1^{p_U}(c) \geq \tilde{M}_2(q(c))$ by Lemma 9. For $K > \overline{K}$, $u(t) = K$ for a maximal interval $[\underline{t}, c]$ is not possible as Lemma 10 requires $M_1^{p_U}(\underline{t}) \leq \tilde{M}_2(q(\underline{t}))$. Hence $u(t) \notin \{0, K\}$ for a set of types with positive measure, arbitrarily close to c. By Lemma 8 and continuity, the third equality follows for c. □

Lemma 16. *Let (U, q, u) be an optimal solution to $\mathcal{P}_C^K$ for $K > \overline{K}$.*

(i) Let $\underline{b} = \min\{q(t) \geq t_2^0\}$. Then there exists $\overline{b} \in [\underline{b}, c]$ such that $u(t) = K$ for $t \in [\underline{b}, \overline{b}]$, and $\tilde{M}_2(q(t)) = \bar{M}_{[\bar{b},1]}^{p_U}(t)$ for $t \in [\bar{b}, c]$. Furthermore, $c = \min\{t | \bar{M}_{[\bar{b},1]}^{p_U}(t) = \overline{v}\}$.

(ii) Let $\bar{t}_1^0 := \max\{t \mid \bar{M}_1^{p_U}(t) \leq 0\}$ and $\bar{t}_1^0 = 0$ if $\bar{M}_1^{p_U}(0) > 0$. Then $\underline{b} \to \bar{t}_1^0$ and $\overline{b} \to \bar{t}_1^0$ as $K \to \infty$.

(iii) For almost every $t < \underline{b}$,

$$u(t) \quad \begin{cases} = 0, & \textit{if } p_q(0) < H^{p_U}(t), \\ \in [0, K], & \textit{if } p_q(0) = H^{p_U}(t), \\ = K, & \textit{if } p_q(0) > H^{p_U}(t). \end{cases}$$

Proof. (iii) follows directly from (B.23)–(B.25) as $q(t) \leq t_2^0$ for $t < \underline{b}$ and hence $m_q(t) = 0$.

If $p_q(0) < H^{p_U}(t)$ for all $t \in [0, 1]$, then $p_q(0) < 0$ and therefore $q(0) = 0$ by the transversality condition. Hence $p_q(0) + m_q(t) < H^{p_U}(t)$, and $q(t) = 0$ for all t, contradicting Lemma 11. Therefore $p_q(0) \geq \min_t H^{p_U}(t)$.

To show (i), we first show that $\tilde{M}_2(q(t)) = \bar{M}_{[\bar{b},c]}^{p_U}(t)$ for all $t \in [\bar{b}, c]$. Three cases have to be considered. To do this we need the following definitions:

$$\underline{p}_q := \begin{cases} \min\{p_q \mid \lambda\{H^{p_U}(t) \leq p_q\}K \geq t_2^0\}, & \text{if } \lambda\{H^{p_U}(t) \leq 0\}K \geq t_2^0, \\ 0, & \text{otherwise,} \end{cases}$$
$$b^{\max} := \max\{b \mid \underline{p}_q \geq H^{p_U}(b)\},$$

where λ denotes the Lebesgue measure on $[0, 1]$.

Case 1: $H^{p_U}(t) > 0$ for all $t > 0$. ($\Rightarrow \underline{p}_q = 0$, $b^{\max} = 0$)
In this case, $q(0) \geq t_2^0$. Otherwise $p_q(0) + m_q(t) < H^{p_U}(t)$ for all $t > 0$. This would imply $q(1) = q(0) < 1$, a contradiction. Suppose $u(t) = K$ for a maximal interval $[\underline{t}, \overline{t}]$. By Lemma 13, $\underline{t} > 0$ would imply $q(\underline{t}) < t_2^0$. Hence $\underline{t} = 0$. Also by Lemma 13, $q(t) \leq x_{p_U}(t)$ for all $t \in [\underline{t}, \overline{t}]$ and hence $q(0) < x_{p_U}(0)$. This implies $p_q(0) + m_q(t) < H^{p_U}(t)$ for t close to zero, contradicting $u(t) = K$. Hence $u(t) < K$ for all $t \in [0, 1]$. This requires $p_q(0) + m_q(t) \leq H(t)$ for all t by (B.23)–(B.25), and

by Reid's Lemma, we have $\tilde{M}_2(q(t)) = M^{p_U}_{[0,c]}(t)$ for all $t \in [0, c]$. With $\underline{b} = \overline{b} = 0$, this shows $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\overline{b}, c]$ in case 1.

Case 2: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) = 0$.
In this case, $q(b^{\max}) = t_2^0$. Suppose to the contrary that $q(b^{\max}) < t_2^0$. This implies $p_q(0) \leq \underline{p}_q$. Hence $p_q(0) + m_q(t) \leq \underline{p}_q < H^{p_U}(t)$ for all $t > b^{\max}$. This is a contradiction to optimality. Next, suppose that $q(b^{\max}) > t_2^0$. This implies $p_q(0) \geq \underline{p}_q$ and therefore $p_q(0) + m_q(b^{\max}) > H^{p_U}(b^{\max})$. Therefore $b^{\max}$ is contained in an interval $[\underline{t}, \overline{t}]$ where $u(t) = K$. By Lemma 13, this is a contradiction. Therefore $q(b^{\max}) = t_2^0$. By (iii) we must have $p_q(0) = \underline{p}_q$ and hence $p_q(0) + m_q(b^{\max}) = \underline{p}_q = H^{p_U}(b^{\max})$. Set $\underline{b} = \overline{b} = b^{\max}$. Lemma 13 also implies that $p_q(0) + m_q(t) \leq H^{p_U}(t)$ for all $t \in [b^{\max}, c]$. Reid's Lemma then implies that $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\underline{b}, c]$ for case 2.

Case 3: $H^{p_U}(t) \leq 0$ for some $t > 0$ and $M_1^{p_U}(b^{\max}) > 0$.
In this case, $q(b^{\max}) > t_2^0$ because otherwise $q(1) = q(b^{\max}) < 1$, which is a contradiction. This implies $\underline{b} < b^{\max}$ and $p_q(0) \geq \underline{p}_q$. Since $p_q(0) \geq \underline{p}_q$, $p_q(0) + m_q(b^{\max}) > H(b^{\max}) = \underline{p}_q$. Hence $b^{\max}$ is in the interior of a maximal interval $[\underline{t}, \overline{t}]$ such that $u(t) = K$ for all $t \in [\underline{t}, \overline{t}]$. By Lemma 13, $q(\underline{t}) < t_2^0$. This implies that $\underline{b} \in (\underline{t}, b^{\max})$. By Lemma 10, $p_q(0) + m_q(\overline{t}) = H(\overline{t})$ and by Lemma 13, $p_q(0) + m_q(t) \leq H(t)$, for $t \in [\overline{t}, c]$. Hence, we can set $\overline{b} = \overline{t}$ and have thus shown $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},c]}(t)$ for all $t \in [\overline{b}, c]$ for case 3.

Claim: $\tilde{M}_2(q(t)) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\overline{b}, c]$.
Note that $\bar{M}^{p_U}_{[\bar{b},1]}(c) \leq \bar{M}^{p_U}_{[\bar{b},c]}(c)$. To show the converse, note that as q is constant on $[c, 1]$, $p_q(0) + m_q(t) + \mu[0, t) \leq H^{p_U}(t)$ for a.e. $t \geq c$. This implies

$$p_q(0) + m_q(c) + (t - c)\overline{v} + \mu[c, t) \leq H^{p_U}(c) + \int_c^t M_1^{p_U}(s)ds,$$

$$\Leftrightarrow \qquad \int_c^t M_1^{p_U}(s)ds \geq \overline{v}(t - c) + \mu[c, t). \tag{B.30}$$

If $\bar{M}^{p_U}_{[\bar{b},1]}(c) < \overline{v}$, then $\int_c^t M_1^{p_U}(s)ds = H^{p_U}(t) - H^{p_U}(c) \leq H^{p_U}(t) - \bar{H}^{p_U}(c) < (t - c)\overline{v}$ for some $t > c$. This would contradict (B.30) so we must have $\bar{M}^{p_U}_{[\bar{b},1]}(c) \geq \overline{v} = \bar{M}^{p_U}_{[\bar{b},c]}(c)$. If $\bar{M}^{p_U}_{[\bar{b},c]}(c) = \bar{M}^{p_U}_{[\bar{b},1]}(c)$ we must have $\bar{M}^{p_U}_{[\bar{b},c]}(t) = \bar{M}^{p_U}_{[\bar{b},1]}(t)$ for all $t \in [\overline{b}, c]$. This proves the claim and $c = \min\{t \mid \bar{M}^{p_U}_{[\bar{b},1]}(t) = \overline{v}\}$ follows immediately. Hence we have shown (i).

It remains to show (ii): $\underline{p}_q \to \min_{t\in[0,1]} H^{p_U}(t)$ as $K \to \infty$. This implies that $b^{\max} \to \overline{t}_1^0$. Furthermore $\overline{b} \geq b^{\max} \geq \underline{b}$ and $\underline{b} - \overline{b} < \frac{1}{K}$. Hence $\underline{b} \to \overline{t}_1^0$ and $\overline{b} \to \overline{t}_1^0$ as $K \to \infty$. □

Now we can turn to the limiting solution as $K \to \infty$.

Proof of Theorem 5. The reduced form of $\bar{x}_i$ as defined in (6.3) is

$$\bar{q}_1(v_1, 2) = \begin{cases} 0, & \text{if } \bar{J}_1^{p_U}(v_1) < 0 \\ \underline{x}_1^0 F_2(v_2^0), & \text{if } \bar{J}_1^{p_U}(v_1) = 0 \\ F_2(J_2^{-1}(\bar{J}_1^{p_U}(v_1)), & \text{if } 0 < \bar{J}_1^{p_U}(v_1) \le \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(v_2, 1) = \begin{cases} 0, & \text{if } J_2(v_2) < 0, \\ F_1((\bar{J}_1^{p_U})^{-1}(J_2(v_2))), & \text{otherwise.} \end{cases}$$

Changing variables, we have

$$\bar{q}_1(t) = \begin{cases} 0, & \text{if } t < \underline{t}_1^0, \\ \underline{x}_1^0 t_2^0, & \text{if } t \in [\underline{t}_1^0, \overline{t}_1^0], \\ M_2^{-1}(\bar{M}_1^{p_U}(t)), & \text{if } 0 < \bar{M}_1^{p_U}(t) \le \overline{v}, \\ 1, & \text{otherwise,} \end{cases}$$

$$\bar{q}_2(t) = \begin{cases} 0, & \text{if } M_2(t) < 0, \\ (\bar{M}_1^{p_U})^{-1}(M_2(t)), & \text{otherwise,} \end{cases}$$

where $\underline{t}_1^0 = \min\{t | \bar{M}_1^{p_U}(t) \ge 0\}$.

Obviously, $\bar{q}_2(t) = \bar{q}_1^{-1}(t)$ if $t \ge t_2^0$ and $\bar{q}_2(t) = 0$ otherwise. Therefore, by Lemma 4, we only have to show optimality of $\bar{q}_1$. Let (q_1^n, q_2^n) be a sequence of optimal solutions of $\mathcal{P}_2^{K_n}$ where $\overline{K} < K_n \to \infty$ as $n \to \infty$. Denote the adjoint variables in these solutions by p_U^n and p_q^n, respectively, and let (q_1, q_2) be the a.e.-limit of the sequence. By Theorem 5, (q_1, q_2) is an optimal solution. We will show that $(\bar{q}_1, \bar{q}_2)$ yields the same expected revenue as the limit of any such sequence. Since $\bar{M}^{p_U}_{[\overline{t}_1^0, 1]}(t) = \bar{M}_1^{p_U}(t)$ for all $t \in [\overline{t}_1^0, 1]$, Lemma 16 implies that $q_1(t) = \bar{q}_1(t)$ for $t > \overline{t}_1^0$ where $p_U = \lim_{n\to\infty} p_U^n$.

Next we consider the limiting solution for $t < \overline{t}_1^0$. If $\underline{t}_0^1 > 0$, then $q_1(0) = 0$ and $u(t) = 0$ for $t \le \underline{t}_1^0$ as for $\bar{q}_1$. Now suppose that $\underline{t}_1^0 < \overline{t}_1^0$.

Claim: If $q_1(t)$ is not constant at $t \in [\underline{t}_1^0, \underline{t}_1^0]$, then $H^{p_U}(t) = \min_\theta H^{p_U}(\theta)$.

Suppose to the contrary that $H^{p_U}(t) > \min_\theta H^{p_U}(\theta)$. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that $H^{p_U}(\tau) > \min_\theta H^{p_U}(\theta) + \delta$ for all $\tau \in (t - \varepsilon, t + \varepsilon)$. Since $p_q^n(0) \to \min_\theta H^{p_U}(\theta)$ for $n \to \infty$, there exists $N > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for all $n > N$, $p_q^n(0) < H^{p_U^n}(\tau)$ for all $\tau \in (t - \varepsilon', t + \varepsilon')$. This implies that q_1^n is constant on $(t - \varepsilon', t + \varepsilon')$ for $n > N$, and hence q_1 is constant on $(t - \varepsilon', t + \varepsilon')$, which is a contradiction. This proves the claim.

Now set $\underline{q}_1^0 = \left[(v_1(\bar{t}_1^0) - v_1(\underline{t}_1^0))\right]^{-1} \int_{\underline{t}_1^0}^{\bar{t}_1^0} q_1(s)v_1'(s)ds$ and let $[\underline{t}, \bar{t}]$ be the interval where $q_1(t) = \underline{q}_1^0$ (if $q_1(t) \neq \underline{q}_1^0$ for all t, set $\underline{t} = \bar{t}$ such that $q_1(t) < \underline{q}_1^0$ if $t < \underline{t}$ and $q_1(t) > \underline{q}_1^0$ if $t > \underline{t}$). With this definition, $q_1(t) < \underline{q}_1^0$ for $t < \underline{t}$ and $q_1(t) > \underline{q}_1^0$ for $t > \underline{t}$, and q_1 is not constant at $\underline{t}$ and $\bar{t}$. The claim implies that $[\underline{t}_1^0, \underline{t}]$ and $[\bar{t}, \bar{t}_1^0]$ are unions of intervals $[a, b]$ such that either $M_1^{p_U}(t) = 0$ for all $t \in [a, b]$, or q_1 is constant on $[a, b]$ and $\int_a^b M_1^{p_U}(t)dt = 0$. Hence, setting $q_1(t) = \underline{q}_1^0$ does not change the value of the objective and by definition of $\underline{q}_1^0$, $U_1(1)$ is left unchanged. Since, $\underline{q}_1^0 = \underline{x}_1^0 t_2^0$, the (q_1, q_2) yields the same expected revenue as $(\bar{q}_1, \bar{q_2})$. Uniqueness of p_U and $\underline{x}_1^0$ are obvious.

For the proof of (ii) and (iii) note that π_2 can be written as

$$\pi_2(\bar{U}) = \int_0^1 \left[\bar{x}_{p_{\bar{U}}}(t)M_1(t) + \int_{\bar{x}_{p_{\bar{U}}}(t)}^1 \tilde{M}_2(q)dq \right] dt.$$

We first show that $\pi_2(\bar{U})$ is Lipschitz. For $\bar{U}' > \bar{U}$,

$$\begin{aligned}
\left|\pi_2(\bar{U}') - \pi_2(\bar{U})\right| &= \left| \int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} M_1(t) - \tilde{M}_2(q)dqdt \right|, \\
&\leq \int_0^1 \left| \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}'}}(t)} \underbrace{M_1(t) - \tilde{M}_2(q)}_{|.|\leq M<\infty} dq \right| dt, \\
&\leq M \int_0^1 \bar{x}_{p_{\bar{U}'}}(t) - \bar{x}_{p_{\bar{U}}}(t)dt, \\
&\leq \frac{M}{\underline{v}_1'}(\bar{U}' - \bar{U}),
\end{aligned}$$

where $\underline{v}_1' = \min_{t\in[0,1]} v_1'(t) > 0$ by our assumptions on the type distributions. Next we show that $\pi_2'(\bar{U}) = -p_U$. for almost every $\bar{U}$. For $h > 0$,

$$\begin{aligned}
\frac{1}{h}(\pi_2(\bar{U} + h) - \pi_2(\bar{U})) &= \frac{1}{h} \int_0^1 \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= \frac{1}{h} \int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1(t) - \tilde{M}_2(q)dqdt, \\
&= -p_{\bar{U}} \frac{1}{h} \underbrace{\int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} v_1'(t)dqdt}_{=h} + \\
&\quad + \int_{\underline{t}_1^0(\bar{U}+h)}^{c(\bar{U})} \frac{1}{h} \int_{\bar{x}_{p_{\bar{U}}}(t)}^{\bar{x}_{p_{\bar{U}+h}}(t)} M_1^{p_U}(t) - \tilde{M}_2(q)dqdt.
\end{aligned}$$

A similar expression can be derived for $h < 0$. $\underline{t}_1^0$ and c are are continuous in $\bar{U}$ for almost every $\bar{U}$ (for all $\bar{U}$ if $M_1^{p_U}$ is strictly increasing). Hence, by the Lebesgue differentiation theorem and dominated convergence, for almost every $\bar{U}$ (every $\bar{U}$ if $M_1^{p_U}$ is strictly increasing),

$$\begin{aligned}\pi_2'(\bar{U}) = \lim_{h\to 0}\frac{1}{h}(\pi_2(\bar{U}+h) - \pi_2(\bar{U})) &= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} M_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt,\\ &= -p_{\bar{U}} + \int_{\underline{t}_1^0}^{c} \bar{M}_1^{p_U}(t) - \tilde{M}_2(\bar{x}_{p_{\bar{U}}}(t))dt,\\ &= -p_{\bar{U}}.\end{aligned}$$

Since $\pi_2(\bar{U})$ is Lipschitz continuous it is absolutely continuous and $\pi_2(\bar{U}) = \pi_2(0) - \int_0^{\bar{U}} p_U(s)ds$. Therefore, as $p_U(\bar{U})$ is non-decreasing, π_2 is weakly concave. If $\{t|\bar{M}_1^{p_U}(t) = 0\}$ is a singleton $p_U(\bar{U})$ is strictly increasing an hence π_2 strictly concave. □

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