

# Constrained efficiency in the neoclassical growth model with uninsurable idiosyncratic shocks

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## Abstract

We investigate the welfare properties of the one-sector neoclassic growth model with uninsurable idiosyncratic shocks. We focus on the *constrained efficiency* notion of the general equilibrium literature, and we demonstrate constrained inefficiency for our model. We provide a characterization of constrained efficiency that uses the first-order condition of a constrained planner's problem that points to the margins of relevance for whether capital is too high or too low: the income composition of the (consumption-)poor. We calibrate our benchmark model parameters governing idiosyncratic risks to the U.S. earnings and wealth distribution, and for this distribution the income of the poor is mainly composed of labor earnings. We compute the constrained-efficient allocations—including transition dynamics—for our model economy, and we conclude that the long-run capital stock in a laissez-faire world is not only too low, but much too low. We also show that one can find parameterizations with different qualitative features: in one case, the steady-state capital stock is too high, and in another case no steady state exists.

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# 1 Introduction

In this paper we investigate the welfare properties of the one-sector neoclassical growth model with uninsurable idiosyncratic shocks but precautionary savings. This kind of model was originally developed and analyzed by Bewley (1986), İmrohorođlu (1989), Huggett (1993), and Aiyagari (1994), and it has become a standard workhorse for quantitatively based theoretical analysis of macroeconomics and inequality. The framework is mostly used for positive analysis, but in this paper we analyze its normative properties in some depth. In particular, we address the issue of whether the competitive equilibrium allocations of this model are efficient relative to the allocations that are attainable through the market. That is to say, we study the *constrained efficiency* of the market allocations (borrowing the term from literature on general equilibrium with incomplete markets introduced in Diamond (1967)) in the one-sector neoclassical growth model with uninsurable idiosyncratic shocks. The idea that with incomplete markets a competitive equilibrium allocation may be inefficient even when compared only to allocations attainable through the existing markets goes back to Diamond (1967). Actually, in that paper the competitive equilibria of a simple one-period, one-good stock market economy with multiplicative uncertainty over production are shown to be constrained efficient. However, examples of constrained inefficiency of competitive equilibria were soon provided in Hart (1975), Diamond (1980), Stiglitz (1982), Loong and Zeckhauser (1982), Newbery and Stiglitz (1984), and Greenwald and Stiglitz (1986). In particular, Stiglitz (1982) established that the constrained efficiency result shown in Diamond (1967) depended on his one-good per state of the world assumption, since with more than one good a reallocation of investments and portfolios would have in general an impact on the relative prices and hence (given the incompleteness of markets) on the agents' welfare. In Geanakoplos, Magill, Quinzii, and Dreze (1990) the authors establish that the constrained inefficiency of the competitive equilibria of a 2-period stock market economy with many goods per state of the world is a generic property in the initial endowments.

In contrast with the general equilibrium literature, we address the constrained inefficiency issue in the infinite horizon model of one sector neoclassical growth with uninsurable idiosyncratic shocks.<sup>1</sup> Specifically, we show that if a planner could simply make consumers

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<sup>1</sup>For a study of uninsurable idiosyncratic shocks from an incomplete-markets, general-equilibrium perspective, see Carvajal and Polemarchakis (2005).

save differently, without in any way completing markets or using transfers between agents, i.e., respecting equilibrium budget constraints and competitive price setting, then the planner should do that. This may not come as a surprise per se: prices are endogenous in our economy, and because asset markets are incomplete, agents' influence on prices leaves room for improvement, even in the absence of transfers between agents.<sup>2</sup> However, what is surprising is the direction of the desired improvement, and the quantitative magnitude of the inefficiency. In particular, we find for a calibrated version of the model that the equilibrium capital stock is *too low*, and that it is *much* too low. This challenges the notion that the precautionary-savings model leads to overaccumulation of capital. To us, it is also a powerful example of how the incompleteness of asset markets may lead to drastic constrained inefficiency that is both intuitive and quantitatively plausible.

We first develop the intuition and prove formal results in the context of a finite-horizon setting. In particular, we first look at a two-period model where the generic inefficiency comes out clearly, though in this case there is indeed always capital overaccumulation. The argument is simple. Suppose that consumers face uninsured wage risk in the second period. Rental income, moreover, is not idiosyncratic, given that all agents are initially alike and save the same amount. A lower capital stock will therefore make total individual income less risky for consumers, because it would lower wages and raise capital returns, thus making the part of income that is stochastic smaller. A command for all agents to lower their savings therefore unambiguously improves welfare, and the result is a general one that does not depend on specifics of the utility function or of technology.

In a three-period model, it is already clear, however, that there may be capital underaccumulation in period 2. This is because at that point, those who had lucky labor outcomes will (given that period- $t$  consumption is a normal good) save more, so if aggregate saving in the second period is induced to fall, the higher return to capital in the third period will help those who were lucky in period two and hence be worse from an ex-ante risk perspective. Whether this channel is more relevant for welfare than the effect via wages, which works in the opposite direction, is a quantitative matter, and we use an infinite-horizon model with a parameterization that matches key inequality statistics to examine this issue carefully. The three-period model also makes clear that the constrained optimum may call for different

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<sup>2</sup>Though we will not use the term here, this kind of effect is sometimes referred to as a “pecuniary externality”.

distortions to the savings of different consumers; we later show that these effects can be important both quantitatively and qualitatively in the infinite-horizon version of the model.

Our study of the infinite-horizon setting is mainly focused on steady states: we assume that the constrained optimum involves convergence to a steady state, and we then numerically examine what such a steady state looks like.<sup>3</sup> This analysis is based on a functional first-order condition that is a necessary condition of the constrained-efficiency planning problem. This first-order condition is, to our knowledge, new, and it is one of the key analytical tools put forth in this paper. Our central finding is that whether there is over- or under-accumulation of capital depends crucially on the *factor composition of the income of the poor agents*. If the poor (consumption-poor) agents have labor-intensive income, then the constrained-efficient allocation involves a larger stock of capital than the market economy delivers by itself. If instead the consumption-poor agents have capital-intensive income, the reverse result holds true. The calibration we employ in the benchmark economy, however, insists on matching central features of the inequality observed in U.S. data—since these are the key features to calibrate to in this kind of analysis—and it delivers a clear message: the consumption-poor are mainly wealth-poor, and hence the planner should increase the capital stock, so that steady-state wages rise. Our calibration is based on Castañeda, Díaz-Giménez, and Ríos-Rull (2003).

We illustrate that other results may obtain if different parameterizations are entertained. One of these involves a case where the consumption-poor agents have capital-intensive income—here risk can be interpreted as unemployment (and not wage) risk—and here we show that the capital stock should be reduced somewhat from a constrained-efficiency perspective. We then look at the parameterization originally used by Aiyagari (1994), which delivers too little wage and wealth inequality, and we find that a constrained-efficient steady state does not exist. Here, we use numerical techniques to argue that the constrained optimum involves convergence in the total capital stock (to a higher value) and ever-increasing wealth inequality: there is increasing returns to saving from a constrained-efficiency perspective.

An aspect of our results worth emphasizing is that the associated full-insurance, or “first-best”, allocation involves *less* long-run capital accumulation than does the laissez-faire out-

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<sup>3</sup>Thus, we look at a “modified Golden Rule”.

come. This means that the in-between concept of constrained optimality—which delivers an in-between ex-ante utility outcome and hence more effective insurance than does the laissez-faire economy—demands long-run capital accumulation that is far from in-between. Instead, the constrained optimum finds capital accumulation a convenient vehicle (in the absence of direct insurance transfers) for achieving better insurance indirectly. In this context, our quantitative finding also contrasts the statements of Aiyagari (1995)<sup>4</sup> and Aiyagari and McGrattan (1995)<sup>5</sup> that various forms of fiscal policy (taxes on capital/government debt) should be used because there is too much capital in a laissez-faire equilibrium. The arguments there are of the standard precautionary-savings nature, and they rely on assuming a form of redistribution (via public goods) or restrictions to proportional taxation. Also, in the present paper the planner has access to state-contingent taxes; we leave restrictions to taxes that are not state-contingent to future work.

Section 2 describes the finite-horizon model and analyzes constrained efficiency in this economy. Section 3 describes the model with an infinite horizon and describes laissez-faire equilibria. The associated constrained-efficiency planning problem is then described in Section 4 and the central first-order condition is derived. Section 4.3 briefly discusses a market implementation of the optimal policy, while Section 5 carries out the quantitative analysis for our calibrated infinite-horizon model and for two alternative parameterizations. Section 6 concludes.

## 2 The mechanisms: illustration using a finite-horizon model

Though our main aim is a quantitative evaluation of efficiency properties of the typical long- or infinite-horizon incomplete-markets economy used in the recent macroeconomic literature, the nature of the mechanisms we wish to point to can be discussed within the context of finite-horizon economies. In the present section we therefore first consider a 2-period model, where we will demonstrate a constrained-inefficiency result—involving *over*-accumulation of capital—that holds quite generally and that has a natural interpretation. There is another mechanism in the infinite-horizon setting that leads to *under*-accumulation of capital, how-

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<sup>4</sup>This paper has a theorem that says if there is a stationary allocation in the Ramsey problem then it has to be associated to  $K^*$ ,  $f_K(K^*, 1)\beta = 1$  which requires positive capital income taxation in the limit. It also provides some example numbers for his favorite calibration and a variety of tax levels.

<sup>5</sup>This paper compares quantitatively (using the type of calibration of Aiyagari (1994)) average utilities across steady states that differ in the level of government debt.

ever, and it cannot be studied with a two-period setting alone. Therefore, we end the present section with a short discussion of how this mechanism would enter in an economy that lasts for 3 or more periods.

## 2.1 A 2-period model

Consider an economy with a continuum (measure 1) of ex-ante identical consumers, each living for two periods. The consumers have time-additive, von Neumann-Morgenstern utility functions with a twice continuously differentiable, strictly increasing and strictly concave period utility function  $u$  and discount factor  $\beta$ . In the first period, period 1, each agent is endowed with  $y$  units of output which can be either consumed,  $c$ , or invested,  $k$ . In period two, consumers receive income from the capital they saved in period 1 and from working. The labor income of any given individual is random. In particular, the labor endowment can be either high or low, and it is independent across agents. We denote the period-2 labor endowments  $e_1$  and  $e_2$ , with  $0 < e_1 < e_2$ ; the probability that any agent's labor endowment is  $e_1$  is  $\pi$ . Due to the independence of shocks across consumers, a law of large numbers operates so that also the *fraction* of agents with  $e_1$  is  $\pi$ . That is, there is no uncertainty about the period-2 labor endowment: the supply of labor is constant at  $L = \pi e_1 + (1 - \pi)e_2$ .

In the second period, output comes from production using capital and labor and a constant-returns-to-scale neoclassical production function  $f$ . Since all agents face the same maximization problem, and since this is a problem with a strictly concave objective and a linear constraint set, they will all make identical choices. Let the implied equilibrium choice of capital be  $K$  (per consumer, and in the aggregate). Then the output in period 2 is known to be  $f(K, L)$ . Output is produced by perfectly competitive firms in our equilibrium: they sell the output to consumers and rent the capital and the labor services from the same consumers at rates  $r$  and  $w$ , respectively. In equilibrium, thus,  $r$  and  $w$  will be set to equal the marginal products of the inputs; in particular, they will be deterministic. This means that in period 1, each consumer will see his capital income in period 2 as deterministic and equal to  $rK$ , whereas his labor income is random and equal to  $w e$ .

It is a maintained assumption in our analysis that consumers can only save using capital; in particular, there is no pure insurance instrument available for reducing the idiosyncratic risk, so the only way of influencing the risk is through “precautionary savings”.

Given the above, we have

**Definition 1.** A competitive equilibrium is a vector  $(K, r, w)$  such that (i)  $K$  solves

$$\max_{k \in [0, y]} u(y - k) + \beta (\pi u(rk + we_1) + (1 - \pi)u(rk + we_2))$$

and (ii)  $r = f_k(K, L)$  and  $w = f_l(K, L)$ , with  $L = \pi e_1 + (1 - \pi)e_2$ .

It is straightforward to show that an equilibrium with  $K \in (0, y)$  exists under suitable (e.g., INADA) conditions on  $u$  and  $f$ .

### Can the market allocation be improved upon? The notion of constrained efficiency

In this economy, agents really only make one choice. Following the incomplete-markets general-equilibrium literature, we discuss the efficiency properties of the equilibrium in terms of whether this one choice could be made in a better way: can it be made so as to improve on equilibrium utility? Formally, we call the equilibrium *constrained efficient* if there is no level of saving  $\hat{K}$  such that, given competitive pricing of inputs in period 2, the utility of the consumer is higher than under the competitive equilibrium. That is, the equilibrium  $(K, r, w)$  we consider is efficient if there is no  $\hat{K} \in [0, y]$  such that

$$u(y - \hat{K}) + \beta \left( \pi u(f_k(\hat{K}, L)\hat{K} + f_l(\hat{K}, L)e_1) + (1 - \pi)u(f_k(\hat{K}, L)\hat{K} + f_l(\hat{K}, L)e_2) \right) > \\ u(y - K) + \beta (\pi u(f_k(K, L)K + f_l(K, L)e_1) + (1 - \pi)u(f_k(K, L)K + f_l(K, L)e_2)).$$

The question, thus, is whether a fictitious planner can improve on the allocation by simply commanding a different savings level for the representative consumer, while respecting all budget constraints of agents and letting firms operate freely under perfect competition? In particular, the fictitious planner is not allowed to “complete markets” or in any way transfer goods between lucky and unlucky consumers: the only insurance asset is still capital.

**The market outcome is constrained inefficient: the formal argument** In this economy, whether it is possible to improve on the market allocation can be seen by considering the impact of a small variation  $dK$  of the aggregate capital. Differentiating the indirect utility one obtains

$$dU = -u_c(y - K)dK + \beta (\pi u_c(rK + we_1)dC_1 + (1 - \pi)u_c(rK + we_2)dC_2),$$

where

$$\begin{aligned} dC_1 &= rdK + Kdr + e_1dw \\ dC_2 &= rdK + Kdr + e_2dw. \end{aligned}$$

The individual's first-order condition for savings reads

$$u_c(y - K) = \beta (\pi u_c(rK + we_1) + (1 - \pi)u_c(rK + we_2)) r.$$

This condition can be used to simplify the above expression, and it will lead many of the effects of increasing capital to vanish. We thus obtain

$$\begin{aligned} dU &= \beta \left( (\pi u_c(rK + we_1) + (1 - \pi)u_c(rK + we_2)) Kdr \right. \\ &\quad \left. + (\pi u_c(rK + we_1)e_1 + (1 - \pi)u_c(rK + we_2)e_2) dw \right), \end{aligned}$$

so that we see that any effect of a marginal change of savings away from the competitive equilibrium has to operate through its effect on factor prices. The cancellations, of course, are just a result of the envelope theorem.

As for how factor prices are affected by capital, we note that

$$\begin{aligned} dr &= f_{KK}(K, L)dK \\ dw &= f_{KL}(K, L)dK \end{aligned}$$

so that

$$\begin{aligned} dU &= \beta \left( \pi u_c(rK + we_1)(Kf_{KK}(K, L) + e_1f_{KL}(K, L)) \right. \\ &\quad \left. + (1 - \pi)u_c(rK + we_2)(Kf_{KK}(K, L) + e_2f_{KL}(K, L)) \right) dK. \end{aligned}$$

Now note that because  $f$  is homogeneous of degree 1,  $Kf_{KK}(K, L) + Lf_{KL}(K, L) = 0$  and therefore

$$dU = \beta \left( \pi u_c(rK + we_1) \left(1 - \frac{e_1}{L}\right) + (1 - \pi)u_c(rK + we_2) \left(1 - \frac{e_2}{L}\right) \right) f_{KK}KdK.$$

Letting  $\hat{\pi} \equiv \pi(e_1/L)$ , we note that  $1 - \hat{\pi} = (1 - \pi)(e_2/L)$ , so that the expression within



parenthesis above becomes

$$\begin{aligned} & \pi u_c(rK + we_1) \left(1 - \frac{e_1}{L}\right) + (1 - \pi) u_c(rK + we_2) \left(1 - \frac{e_2}{L}\right) = \\ & \pi u_c(rK + we_1) + (1 - \pi) u_c(rK + we_2) - (\hat{\pi} u_c(rK + we_1) + (1 - \hat{\pi}) u_c(rK + we_2)) \\ & = (\pi - \hat{\pi})(u_c(rK + we_1) - u_c(rK + we_2)) > 0 \end{aligned}$$

because  $e_2 > e_1$  and  $\pi - \hat{\pi} = \pi(1 - (e_1/L)) > 0$ . Therefore, for  $dK > 0$ , since  $f_{KK} < 0$ ,

$$dU = \beta(\pi - \hat{\pi})(u_c(rK + we_1) - u_c(rK + we_2))f_{KK}KdK < 0.$$

We conclude, in other words, that the equilibrium is *constrained inefficient*. As is clear from the analysis, the key assumptions behind the result is that  $u$  is strictly concave and that  $f$  has a strictly decreasing marginal product of capital.

More specifically, the level of capital in the laissez-faire equilibrium is *too high*: a higher utility is obtained if all consumers save a little less in period 1. The intuitive reason for the overaccumulation of capital is as follows. More capital savings raises wages and lowers rental rates. The only source of market failure in this economy is the incomplete insurance. A small decrease in  $K$  from the equilibrium level thus lowers  $w$  and raises  $r$ , thereby scaling down the part of the consumer's income that is stochastic and scaling up the part that is deterministic: the amount of risk the consumer is exposed to is now smaller. Given that there is no direct insurance for this risk, this amounts to an improvement. The "distortion" on the agents' savings by moving savings away from the competitive-equilibrium level for given prices is of a second-order magnitude, and thus the manipulation of prices so as to lower the de-facto risk dominates.

Market incompleteness is of course key to our finding of constrained inefficiency: unlike in the complete-markets case, prices are not optimally set here and agents' influence on prices should therefore be taken into account when making individual choices. An improvement on the competitive outcome thus requires taking an aggregate, "planning" perspective.

## 2.2 More periods

In the two-period model, apparently, our overaccumulation result obtains rather generally. Our main focus here, however, is longer-lived economies; indeed, the next section of the

paper examines the typical macroeconomic setting with an infinite time horizon. But before we discuss the infinite-horizon model in detail, what additional mechanisms appear in longer-horizon models? In this section we will use a three-period model, where there are neoclassical production and idiosyncratic labor endowment shocks both in periods 2 and 3, to illustrate how the analysis of constrained efficiency changes with more periods.

**Over- or under-accumulation?** Following the analysis of the 2-period model, one can study the optimal savings decision in the intermediate period 2. Now there are two kinds of agents to consider: those who had a high labor endowment realization in this period, and those who had a low one. Specifically, one can look at whether an increase in the aggregate capital stock carried from period 2 into period 3 will raise the utility of consumer  $i$ , where  $i$  refers to the labor endowment realization in period 2. As before, utility is only influenced through the price effects, and the impact on consumer  $i$ 's equilibrium present-value utility can be derived in a straightforward manner along the lines of the above analysis. This impact is

$$dU_i = \beta^2 \left( \pi u_c(rK_i + we_1) \left( \frac{K_i}{K} - \frac{e_1}{L} \right) + (1 - \pi) u_c(rK_i + we_2) \left( \frac{K_i}{K} - \frac{e_2}{L} \right) \right) f_{KK}(K) K dK.$$

where for simplicity the notation has been maintained— $K$ ,  $L$ ,  $r$  etc. refer to third-period values—and the idiosyncratic shocks have been assumed to be iid. Moreover,  $K_i$  refers to the savings of the consumer with an  $i$  shock, where now  $K_1 < K_2$  is expected, since type-2 consumers are richer ex post. Inspecting this expression, we see that a decrease in capital—which is unambiguously beneficial in a 2-period economy—is a plus for type-2 agents and a minus for type-1 agents, since  $K_2 > K > K_1$ . Moreover, note that the planner maximizes period-0 utility, which makes the de-facto weight on type-1 agents larger, because their marginal utilities in the third period are lower due to their lower savings:  $u_c(rK_1 + we_j) > u_c(rK_2 + we_j)$  for  $j = 1, 2$ . This means that we have uncovered a reason why a decrease in savings may be detrimental: *the implied increase in the rental rate in period 3 will help those who are lucky in period 2*, thus making the lack-of-insurance problem more severe. This effect, as it turns out, will under some calibrations be more important quantitatively than the direct effect that is present in the 2-period model. In the next section, we will discuss what features of the economy are key for whether aggregate equilibrium savings are too high or too low. In that section we will, moreover, derive a first-condition for savings that comes from maximizing consumer welfare subject to the given market structure

and that allows us to talk also about the “best” level of savings. That first-order condition in particular summarizes the conflicting effects on savings.

**Who should save?** Another interesting aspect of our efficiency analysis—that will also play a central role in one of our examples in the next section—is that there are nontrivial distributional implications for what a planner should do. In particular, the constraints on what the planner can do to improve utility do not rule out choosing the savings of the two consumers with different labor endowment realizations in period 2 separately. In the above discussion, we increased aggregate capital and it was implicit that all agents’ savings were increased by equal amounts. Would the planner command lucky consumers to save more or less than unlucky consumers? The general idea here is that (i) agents that are “lucky” are agents with ex-post low marginal utility and therefore (ii) agents with low marginal utility have lower marginal costs of increased savings (and lower marginal benefits of decreased savings). As an example, if the pecuniary externality of savings that exists in this model is positive—so that more aggregate saving is desirable—the planner would want to make ex-post lucky agents increase their savings more than unlucky agents. Hence, we would obtain an argument for increased differences in asset inequality. Of course, this increase in asset inequality is no redistribution from unlucky to lucky consumers, and indeed consumption early on becomes less dispersed, because the whole point is that ex-post utilities should become less dispersed.

### 3 The infinite-horizon economy

We now study the infinite-horizon model. In essence, it is a long-horizon version of the economy described above.

#### 3.1 The economy and recursive competitive equilibrium

As above, we look at a continuum of agents subject to idiosyncratic shocks  $e_i \in E$ , where  $E \equiv \{e_1, \dots, e_i, \dots, e_I\}$ , that are i.i.d. across agents and that follow a Markov process with transition matrix  $\pi_{i,j}$ . These shocks are the amount of efficient units of labor that agents have each period. Agents have standard preferences: an expected discounted sum of a strictly increasing and strictly concave utility function, i.e.,  $\mathbb{E}_0 \{\sum_t \beta^t u(c_t)\}$ . Agents do not have access to state-contingent contracts but can only accumulate assets in the form of

real capital; we denote it  $a$ . There is a lower bound on assets:  $\underline{a}$ .<sup>6</sup> We assume a borrowing constraint that prevents these assets from being negative. Moreover, we assume a very large upper bound on assets,  $\bar{a}$ , implying  $a \in A = [0, \bar{a}]$ . As in the previous section, these assets are rented by competitive firms each period and used for production purposes according to a constant-returns-to-scale neoclassical production function  $f$  that uses capital and efficient units of labor. Capital accumulation is assumed to follow a geometric structure: a fraction  $\delta$  of the capital stock depreciates from one period to the next.<sup>7</sup>

The nature of the budget constraint that agents face is thus

$$c + a' = a(1 + r) + e w \tag{1}$$

where we use primes to denote next period's values and where  $r$  and  $w$  are the rental prices of capital and labor that have yet to be determined.

Individual agents are indexed by the pair  $\{e, a\}$  that describes their labor endowment and wealth, respectively. The state of the economy can be summarized by means of a probability measure  $x$  over the Borel sets of compact set  $S = E \times A$ . In this context, aggregate amounts of factors of production and their rental prices are

$$K = \int_S a \, dx, \quad r = f_K(K, L) - \delta, \tag{2}$$

$$L = \int_S e \, dx, \quad w = f_L(K, L), \tag{3}$$

and we write  $r(x)$  and  $w(x)$ . On occasion we also write  $r(K)$  and  $w(K)$  since aggregate labor  $L$  is constant due to the law of large numbers.

In this economy the aggregate state variable is the distribution of agents over labor earnings and wealth,  $x$ , which agents have to know in order to compute prices.<sup>8</sup> We write

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<sup>6</sup>This lower bound may arise from the existence of a solvency constraint that requires that agents are always able to pay back their debt or from an explicit borrowing constraint. The latter is used in the popular case that restricts assets to be nonnegative.

<sup>7</sup>Note that the assumptions in the previous section can be thought of as assuming 100% depreciation, or that alternatively  $f$  was defined to include undepreciated capital.

<sup>8</sup>While prices today can be known just from today's aggregate capital, future prices cannot be known from today's aggregate capital because decision rules are not linear. Hence the distribution is the appropriate state variable. See Krusell and Smith (1997), Krusell and Smith (1998), or Ríos-Rull (1998) for more elaborate discussion.

$x' = H(x)$  to describe the law of motion of the distribution. Then, the agent's problem is

$$v(x, e, a) = \max_{\substack{c \geq 0 \\ a' \in A}} u(c) + \beta \sum_{e'} \pi_{e,e'} v(x', e', a') \quad \text{s.t.} \quad (4)$$

$$c + a' = a [1 + r(x)] + e w(x), \quad (5)$$

$$x' = H(x), \quad (6)$$

with solution  $a' = h(x, e, a)$ . An important feature of this problem is the requirement that the agent's assets lie in compact set  $A$ .

We now turn to the construction of an aggregate law of motion of the economy. Using decision rule  $h$  and transition matrix  $\pi$ , we construct an individual transition process. Let  $B \in S$  be a Borel set. Define  $Q$  by

$$Q(x, e, a, B; h) = \sum_{e' \in B_e} \pi_{ee'} \chi_{h(x,e,a) \in B_a} \quad (7)$$

where  $\chi$  is the indicator function. It is easy to see that  $Q$  is indeed a transition function. We now define the updating operator  $T(x, Q)$  that yields tomorrow's distribution given today's:

$$x'(B) = T(x, Q)(B) = \int_S Q(x, e, a, B; h) dx \quad (8)$$

An equilibrium requires that agents expectations are correct. Formally,

**Definition 2.** *A recursive competitive equilibrium is a pair of functions  $h$  and  $H$  such that  $h$  solves problem (4) given  $H$  and that  $H(x) = T(x, Q(\cdot; h))$ .*

A *steady state* for this economy is a distribution  $\tilde{x}$  such that  $\tilde{x} = T(\tilde{x}, Q)$ . Steady states have the property that the interest rate is lower than the rate of time preference, or that the aggregate capital stock is higher than that of an economy with perfect markets or no shocks (for a discussion and a proof of this result, see Huggett (1997)). The interpretation of this result is one of *precautionary savings*: savings play dual roles here, by not just allowing intertemporal smoothing but also some (limited) amount of smoothing across states.

### 3.2 Characterization: first-order conditions and steady-state capital

In a recursive competitive equilibrium, the consumer's first-order condition for savings is

$$u_c(a [1 + r(x)] + e w(x) - a') \geq \beta \sum_{e'} \pi_{e,e'} v_3(x', e', a') \quad (9)$$

with equality if  $a' > \underline{a}$ . The envelope condition is

$$v_3(x, e, a) = [1 + r(x)] u_c(a [1 + r(x)] + e w(x) - a'). \quad (10)$$

Combining the two conditions and using the economy's law of motion and the agent's decision rule, we obtain

$$u_c(x, e, a, h(x, e, a)) \geq \beta [1 + r(H(x))] \sum_{e'} \pi_{e,e'} u_c(H(x), e', h(x, e, a), h[H(x), e', h(x, e, a)]), \quad (11)$$

which can be rewritten compactly as

$$u_c \geq \beta [1 + r(H(x))] \sum_{e'} \pi_{e,e'} u'_c. \quad (12)$$

Steady states can be readily found by finding a fixed point of an aggregate steady-state capital demand function, which depends on the interest rate which in turn is given by the marginal productivity of capital (see below). Let  $h^m(e, a; r)$  be the decision rule implied by a constant interest rate  $r$  (and associated wage  $w$ ). It solves

$$u_c[a(1+r) + e w - h^m(e, a; r)] \geq \beta \sum_{e'} \pi_{e,e'} u_c[h^m(e, a; r) (1+r) + e w - h^m[e', h^m(e, a; r); r]], \quad (13)$$

with equality if  $h^m > \underline{a}$ .

Let the stationary aggregate capital implied by  $h^m(\cdot, \cdot; r)$  be  $K(r)$ , a continuous function of  $r$  (Ríos-Rull (1998)). A steady state is therefore characterized by a  $\bar{K}$  and a rate of return  $\bar{r}$  such that given  $\bar{r}$ , aggregate capital is  $\bar{K}$ , i.e.,  $\bar{K} = K(\bar{r})$ , and  $\bar{r}$  is the marginal productivity of capital implied by  $\bar{K}$ , i.e.  $\bar{r} = f_k(\bar{K}, L)$ . In a steady state, where the discount rate exceeds the interest rate, there is an upper bound to the assets that agents hold (see Huggett (1997))

so the upper bound assumed to exist is not exogenously imposed but generated by the model.

## 4 Constrained-optimal allocations in the infinite-horizon economy

Though our focus will be on optimal steady states, we have in mind an initial condition like the one in the 2-sector economy: all agents start out alike. I.e., we have in mind a pure insurance problem where the objective of the planner, beside the standard objective of accumulating capital, is to make sure that agents save so as to minimize the losses due to missing insurance markets. Hence a possible interpretation of the planner’s concerns is that it takes the effects of prices explicitly into account. Under complete markets, the effects of consumers’ savings on prices could be ignored, because of the first welfare theorem, but here they cannot.

Our main characterization result—the first-order condition below—is derived using a variational approach and thus based on a sequential formulation of the constrained-efficiency problem. However, for descriptive purposes, we use a recursive formulation of this planning problem in the main text. It reads

$$\Omega(x) = \max_{y(e,a) \in A} \int_S u[a(1+r(K)) + ew(K) - y(e,a)] dx + \beta \Omega(x') \quad (14)$$

$$\text{s.t.} \quad x' = T(x, Q(\cdot; y)), \quad K = \int_S adx. \quad (15)$$

We use the function  $h^*$  to denote the implied decision rule for  $y(e, a)$  at  $x$ :  $a' = h^*(x, e, a)$ .

This recursive program weighs all agents’ utilities equally: it appears “utilitarian”. This specification follows from two assumptions. First, we have in mind an initial condition where all agents are identical: at time 0 they have identical wealth and identical wage status.<sup>9</sup> Second, given that consumers are identical at time 0, we assign equal weights to them, because we wish to study a pure insurance problem and not treat identical consumers differently.<sup>10</sup> Because we have this guidance in choosing weights, we are thus able to make

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<sup>9</sup>With equal wage status for all agents in the initial period, aggregate labor will not be constant over time, but it will transit exogenously and deterministically and, due to the law of large numbers, converge to a constant. For simplicity, in the statement of the recursive program above, we presume that aggregate labor is always constant—since we focus our analysis below on long-run outcomes, this simplification is not restrictive.

<sup>10</sup>It is possible, of course, to imagine different initial conditions. In such cases, however, it would be less

*precise statements* about the nature of constrained-optimal long-run outcomes.

#### 4.1 The first-order condition

For convenience, we will now assume that the distribution  $x$  admits a density.<sup>11</sup> Our key analytical characterization in this paper is the first-order conditions for the planner:

**Proposition 1.** *If the distribution  $x$  admits a density, the first-order necessary conditions of problem (14) can be stated as the following functional equation in the decision rule  $h^*$ : for all  $\{e, a\} \in S$ ,*

$$\begin{aligned}
& u_c(a [1 + r(K)] + ew(K) - h^*(x, e, a)) \geq \\
& \beta [1 + r(K')] \sum_{e'} \pi_{e,e'} u_c(h^*(x, e, a) [1 + r(K')] + e'w(K') - h^*(x', e', h^*(x, e, a))) + \\
& \beta \int_S [e' f_{LK}(K', L) + a' f_{KK}(K', L)] u_c(a' [1 + r(K')] + e'w(K') - h^*(x', e', a')) dx'
\end{aligned} \tag{16}$$

where the inequality becomes equality if  $h^*(x, e, a) > \underline{a}$ .

*Proof.* See Appendix A. □

For later reference, we will define  $Y$  as the implied law of motion of the distribution:  $x' = Y(x) = T(x, Q(\cdot; h^*))$ .

Omitting arguments, we can write the first-order condition compactly as

$$u_c \geq \beta (1 + r') \sum_{e'} \pi_{e,e'} u'_c + \beta \int_S [e' f'_{LK} + a' f'_{KK}] u'_c dx'. \tag{17}$$

This equation is the guide for individual savings at different values for  $(e, a)$ . Thus, it can be compared to equation (12), which characterizes the laissez-faire allocation. We see that the difference between the equations is the third and last term of (17); it was not present in (12) (the other two terms are common). The new term has a number of noteworthy features

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clear how to assign weights to the different agents.

<sup>11</sup>That our characterization is possible to carry out without this assumption was kindly pointed out to us by Iván Werning.



that change incentives in interesting ways and lie behind the quantitative findings that we present in the quantitative part of the paper.

1. This kind of new term *would also appear in finite-horizon models*. In the two-period model, the condition is identical in form to the one above, but with (i)  $h^*$  being the same for all agents in the first period since they all start out the same; and (ii) savings being zero for all agents in the second period.
2. The new term captures the additional *average effect on utility from increased savings that is accomplished through price changes*. It is thus the sum of tomorrow's changes in total income weighted by the marginal utilities of the different agents receiving the income. These weights are higher the lower are the consumption levels of the corresponding consumers.
3. *In a representative-agent special case the new term is zero*, and hence the equilibrium is constrained-optimal (as it should be, given that the first welfare theorem applies in that case). To see this, note that representative-agent model collapses the integral with respect to wealth yielding

$$u_c \geq \beta (1 + r') \sum_{e'} \pi_{e,e'} u'_c + \beta \sum_{e'} \pi_{e,e'} [L' f'_{LK} + K' f'_{KK}] u'_c.$$

But the terms within brackets sum to zero by the Euler theorem since the production function is homogeneous of degree 0.

4. The new term can be *either positive or negative*. Its sign is particularly influenced by the sign of the term in brackets for the high marginal utility agents—the consumption-poor agents—since they receive a higher weight. Since the correlation between income and wealth (and thus consumption) is less than one, which of the two variables determines the relevant notion of poverty depends on the persistence properties of the shocks. Roughly speaking, the optimal plan assigns consumption poverty to those agents who are the likeliest to be hit by the borrowing constraint in the future.
5. If poor agents have labor-intensive income relative to the economy as a whole then the term in brackets is positive for them, because of the Euler theorem, and then with enough poor agents so is the whole new term. Consequently, the sign of the new

term, and thus whether there is too much capital or too little capital in this economy, *depends to a large extent on whether the poor agents' income is labor-intensive or capital-intensive*. If it is labor-intensive, they would benefit from more capital, since it would raise their total income, and the planner would therefore like to have more capital than the market economy.

6. Whether poor people have labor-intensive income or capital-intensive income is a result of the primitives of the model, but in a quantitatively restricted model *these primitives of course have to be selected to match the relevant economic data*. To briefly see this, imagine that  $e$  is i.i.d. and it can take two values, one of these being very unlikely and very small (say, “unemployment”). In such an economy agents save to prevent suffering from low consumption in that state. When the state arrives, thus, labor income is very low while capital income is high in relative terms. As a result, the poor’s income is capital-intensive, and hence the planner would want less capital than what the market allocates. Alternatively, in model economies with a substantial right tail for earnings, wealthy people will mainly be capital rich, and poor people mainly capital poor. Thus, whether the new term is positive or negative is an empirical issue. We discuss this in detail in Section 5, where we both propose a reasonable calibration and illustrate how alternative parameterizations give different results.
7. Finally, the new term is independent of the consumer’s wealth/wage status, and therefore its effect on behavior—its distortionary impact relative to the laissez-faire benchmark—is larger for agents with low marginal utility. In other words, it *influences rich agents more than it influences poor agents*, and therefore it affects the wealth distribution. Thus, it suggests that when capital is too low in equilibrium, the increases in savings will be executed mostly by the rich who, as a consequence, suffer lower consumption. This is, of course, part of a desired imperfect measure to improve insurance. We will see below that this mechanism can be very powerful quantitatively and even lead to ever-widening asset inequality. Conversely, when there is too much capital, the decrease in savings will also be disproportionately borne by the rich; here as well, the rich are forced to a larger deviation from what their savings would be absent government intervention.

## 4.2 The constrained-efficient steady state

We now turn to characterizing the steady state of this economy. A steady state for the planner is a decision rule  $\bar{h}^*$  and an associated distribution  $\bar{x}^*$  such that  $\bar{x}^* = Y(\bar{x}^*)$  and  $\bar{h}^*(e, a) = h^*(\bar{x}^*, e, a)$ . With  $\bar{K}^* \equiv \int_S a d\bar{x}^*$ , a steady state satisfies

$$\begin{aligned}
& u_c(a [1 + r(\bar{K}^*)] + e w(\bar{K}^*) - \bar{h}^*(e, a)) \geq \\
& \beta [1 + r(\bar{K}^*)] \sum_{e'} \pi_{e,e'} u_c(\bar{h}^*(e, a) [1 + r(\bar{K}^*)] + e' w(\bar{K}^*) - \bar{h}^*(e', \bar{h}^*(e, a))) + \\
& \int_S [e' f_{LK}(\bar{K}^*, L) + a' f_{KK}(\bar{K}^*, L)] u_c(a' [1 + r(\bar{K}^*)] + e' w(\bar{K}^*) - \bar{h}^*(e', a')) d\bar{x}^*.
\end{aligned} \tag{18}$$

This is a functional equation that can be solved using standard numerical methods. Note in this context that if the last term is positive, there is no guarantee that there exists an upper bound to individual asset holdings. However, an upper bound can be imposed, and in the numerical simulations one would then verify whether or not it is violated.

It is important to note, in this context, that the object of study in our steady-state analysis is a “modified Golden Rule”, i.e., a long-run outcome that is optimal from the perspective of taking discounting into account. In other words, our constrained-optimal steady state is *not* derived simply from maximizing steady-state utility without regard to initial conditions and the costs or benefits of reaching that steady state. Instead, it answers the question: if the allocation that is ex-ante constrained optimal has the property that there is convergence to a steady state, what are the properties of the implied steady-state distribution?

## 4.3 Implementation of the constrained optimum

The allocation chosen by the planner involves no transfers between agents and is not implementable with mechanisms of an aggregate nature: the planner “instructs” each agent, conditional on asset holding and wage realization, how to save. In order to replicate this command allocation, hence, it is necessary for a government to use taxes and lump-sum

transfers that are type-dependent, so that the requirement of no net transfer among agents can be met. Thus, we can think of this as (i) a distortion to saving, using a tax schedule that is nonlinear, i.e., where marginal rates vary with wealth, and that is also type-dependent, i.e., varying with the wage realization, together with (ii) a lump-sum transfer that ensures that the net transfer is zero.

While the mentioned policies can be readily computed from the solution to the planner's problem, they should not be viewed as a proposal for an actual tax package. The reason is that in general (but not always) they rely on type-specific information, and given this information the government could do even better: it could implement the transfers we do not allow, and thus fully insure agents. A special case in which the constrained-optimal policy would actually correspond to a policy that does not require type-specific information can be found in the two-period example we discussed first. There, the constrained optimum can be implemented with an investment subsidy in the first period. That is, suppose that the government cannot tax or transfer at all in the second period, but that agents can costlessly trade anonymously in the markets for capital and labor in period two. Then the very best allocation that can be achieved with government intervention is the constrained optimum we compute.

It is an open question how more restrictions (or, more generally, other restrictions) on government policy, such as forcing tax schedules to not depend on the wage realization, would change the characterization. The purpose of this paper is the more narrow one of pinning down the exact nature of how agents, by not taking into account how their decisions influence prices, make decisions that are not optimal from the perspective of risk sharing under the kind of market incompleteness considered in Bewley-type economies. We find the questions about implementable policy interesting, of course, but postpone them for future inquiry.

## **5 Quantitative analysis: an economy calibrated to match the U.S. earnings and wealth distribution**

We now turn to the quantitative analysis by specifying various specific model economies. We start in Section 5.1 with the Baseline Model Economy calibrated to have its market allocation steady state replicate the main properties of the earnings and wealth distribution in the U.S.

as reported in recent SCF surveys. In doing this, we follow Castañeda, Díaz-Giménez, and Ríos-Rull (2003) and Diaz, Pijoan-Mas, and Ríos-Rull (2003). We then continue in Section 5.2 with the Unemployment Economy where good and bad shocks are designed to look like employment and unemployment following Krusell and Smith, Jr. (1996). It turns out that these two economies display the exact opposite outcomes, the planner in the baseline economy chooses a much larger steady state amount of capital than the one that obtains via a market mechanism, while the planner wants less capital than the market in the Unemployment Economy. Section 5.3 discusses the reasons for such differences and presents some robustness results that provide further insights in the working of the model. Finally, in Section 5.4 we use the parameterization chosen by Aiyagari (1994) in his seminal paper and we show that in this case the constrained optimum may be inconsistent with a stationary wealth distribution, but we still are able to characterize its properties.

### 5.1 The baseline model economy: replicating the U.S. earnings and wealth distribution

Preferences are of the CRRA form,  $\sum_t \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$ , with the period set to be one year. Production occurs through a standard neoclassical production function  $F(K_t, L_t) = K_t^\theta L_t^{1-\theta}$ .

In the model economy considered by Aiyagari (1994), the coefficient of variation for the earnings distribution is 0.2, with a Gini index of 0.11. The corresponding values of the coefficient of variation and the Gini index for the U.S. economy are 2.65 and 0.61, respectively.<sup>12</sup> This is one reason why the wealth distribution in the Aiyagari parameterization does not at all look like the actual U.S. wealth distribution. In our calibration here, a key requirement is that not only the dispersion of earnings but also that of wealth are realistic, so we need to move away from Aiyagari's parameterization. Though there are several possibilities here, we will focus on a simple way of achieving the goal: we follow the calibration described in Diaz, Pijoan-Mas, and Ríos-Rull (2003), which is a simplified version of that in Castañeda, Díaz-Giménez, and Ríos-Rull (2003). More precisely, a properly chosen 3-state Markov chain allows us to generate a laissez-faire steady state with inequality measures for earnings and wealth quite close to those in the U.S. data. We report the earnings process that we use in Table 1 below.

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<sup>12</sup>The earnings data used by Aiyagari (1994) comes from the PSID which misses the right tail of both the earnings and the wealth distribution. See Budría, Díaz-Gimenez, Quadrini, and Ríos-Rull (2001).

Table 1: **Earnings process**

$e \in \{e_1, e_2, e_3\} =$	{1.00,	5.29,	46.55}
$\pi_{e,e'} =$	$\begin{bmatrix} 0.992 & 0.008 & 0.000 \\ 0.009 & 0.980 & 0.011 \\ 0.000 & 0.083 & 0.917 \end{bmatrix}$		
$\pi^* =$	0.498	0.443	0.059

The table reveals that in order to generate a high Gini coefficient with just three points in the Markov chain, each state must be very different: the labor earnings of the lucky households is almost 50 times those of the unlucky ones. The process for earnings that we use has a Gini index of 0.60.

In addition, the high earnings variability market economy is calibrated so that the steady state of the market economy has an interest rate of 4%. The capital-output ratio is slightly below 3 and the labor share is 0.64, which is accomplished assuming that  $\beta = 0.887$ ,  $\delta = 0.08$ , and  $\theta = 0.36$ . The intertemporal elasticity of substitution is set to 0.5.

The results for the economy calibrated in the manner just described are contained in Table 2.

Table 2: **The steady states for the baseline model economy**

	Deterministic Economy	Market Economy	Planner Economy
Aggregate Assets	1.736	4.017	15.668
Output	1.000	1.353	2.208
Capital-Output ratio	1.736	2.970	7.096
Interest Rate	12.740%	4.123%	-2.927%
Coeff. of Variation of Wealth	0.0	2.562	2.501
Gini Index of Wealth	0.0	0.861	0.864

The first column of the table reports the steady states of the deterministic version of the

economy (the “first best” when all income shocks are perfectly insured), the second column reports the steady state of the incomplete-markets allocation, and the third column shows the steady state of the constrained-efficient allocation. In this table, TFP was normalized so that output in the deterministic version of the model is 1.

The first notable result is that the market economy has large precautionary savings: aggregate wealth is 2.33 times larger than in the economy without shocks. As a result of the additional capital output is 35.5% higher. We see also that the Gini index of the market economy is quite large, 0.853, slightly larger than the 0.803 of the U.S. economy.

The market allocation reveals significantly higher long-run capital accumulation than does the deterministic economy, but the constrained optimum implies an even higher level of capital: it is a whopping 8.5 times higher than in the deterministic economy and even 3.65 times higher than in the market allocation. This is an enormous difference. Moreover, it is a nontrivial finding in that it contradicts the intuition that the “in-between” allocation in an efficiency sense—the allocation improves on laissez faire but is dominated by the first best—should produce an “in-between” amount of long-run capital.

Table 3: **The distribution of wealth in the baseline model economy**

Economy	<i>Quintiles</i>					<i>Top Groups (%)</i>		
	1st	2nd	3rd	4th	5th	90-95%	95-99%	Top 1%
High Earn Vol Market	0.00	0.00	1.45	3.40	95.16	25.38	38.00	14.55
High Earn Vol Planner	0.00	0.00	0.44	3.31	96.25	25.20	35.72	14.54
USA 1998	-0.30	1.30	5.00	12.20	81.70	11.30	23.10	34.70

Table 3 shows the share of wealth held by selected groups in both the market and the constrained-optimal allocation of the model economy as well as in U.S. data. We see that the high earnings inequality in the model economy exaggerates the wealth concentration of the U.S. as measured by the share of wealth of the highest quintile, although it generates a lower share of wealth held by the top 1%, which is a little less than one half of that in the data. This is a compromise that comes from the fact that we are using a very parsimonious income process; not all statistics can be targeted. What Table 3 shows is that the wealth distribution of the market and the constrained-optimal allocations are very similar, save for the enormous difference in total assets held. The similarity in assets holdings can also be

seen in Figure 1.

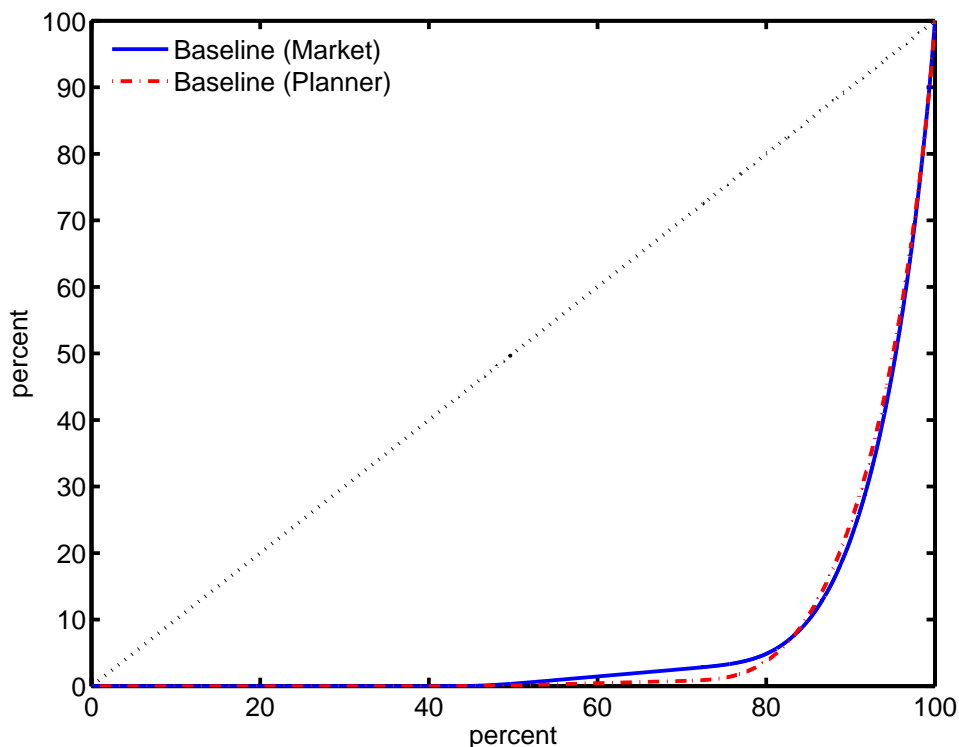


Figure 1: LORENZ CURVE FOR ASSETS: BASELINE ECONOMY

Figure 2 describes key mechanisms in the model; it derives aggregate long-run supply and demand relationships for capital as a function of an interest rate that is assumed constant over time: what total savings would be, and what total demand by firms would be, respectively.

The interest rate associated to the economy with full insurance is indicated with discontinuous lines. The demand for capital is just the schedule for marginal productivity schedule, while the supply of capital is the amount of aggregate asset holdings with respect to the stationary distribution generated by each interest rate. For the constrained-optimal economy, the steady-state supply of capital is based on a calculation where, for each interest rate, the relevant value of the third term of equation (18), denoted  $g$ , is solved for given the stationary distribution. This involves a fixed point: given a guess on  $r$  and  $g$ , one computes a stationary distribution, which delivers updated values for  $r$  and  $g$ .

To summarize, the constrained-efficient allocation in the model economy has much more capital than does the laissez-faire allocation (which itself has much more capital than does the



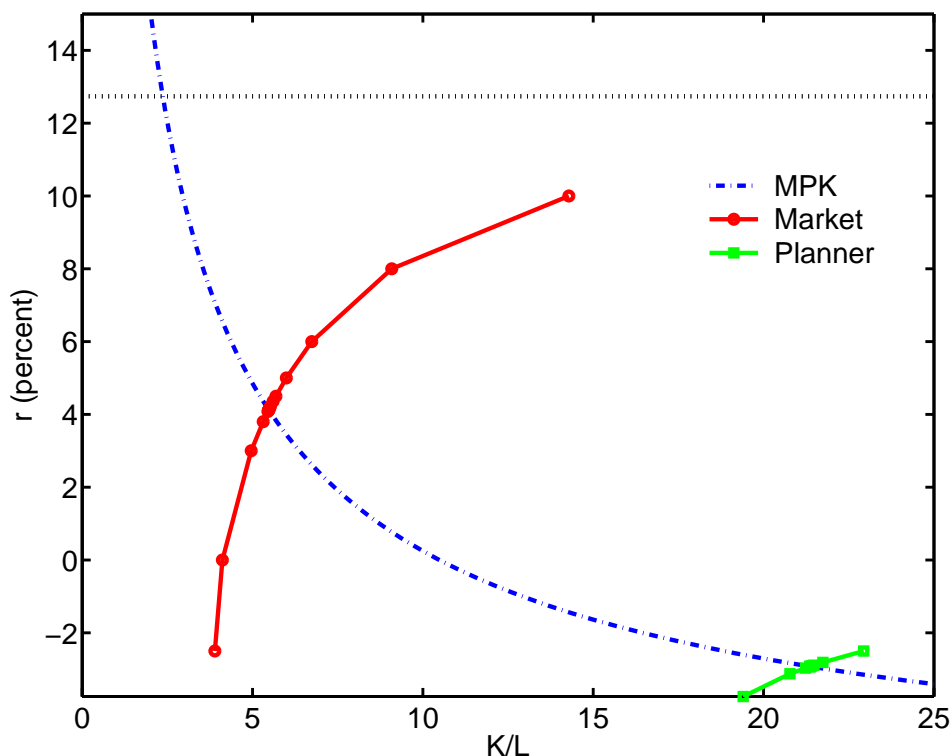


Figure 2: STEADY-STATE CAPITAL SUPPLIES AND DEMANDS FOR THE MARKET AND CONSTRAINED-EFFICIENT ECONOMIES

first-best allocation—due to precautionary savings), despite the fact that the distribution of wealth as measured by the shares owned by the various groups is very similar. Said differently, the precautionary-savings economy generates *too little* capital.

## 5.2 The Unemployment Economy

In the Unemployment Economy the idiosyncratic shock captures unemployment risk rather than wage risk: it can take only two values, with a very low value of unemployment as in Krusell and Smith, Jr. (1996) or Castañeda, Díaz-Giménez, and Ríos-Rull (1998). We calibrate the economy to an unemployment rate of 5% and an average duration of unemployment of 2.6 years. The parameters involved are  $e = [0.01 \ 1.00]$  for the labor endowment (unemployed, employed) and  $\Pi_{1,\cdot} = [0.62 \ 0.38]$  and  $\Pi_{2,\cdot} = [0.02 \ 0.98]$  for the transition matrix. Table 4 displays the steady states of the full-insurance economy, market economy, and the constrained-efficient economy.

The constrained optimum implies a lower long-run level of capital than what is generated

Table 4: **The steady states of the Unemployment model economy**

	Deterministic Economy	Market Economy	Planner Economy
Aggregate Assets	2.959	3.359	3.279
Output	1.000	1.047	1.038
Capital Output ratio	2.959	3.209	3.160
Interest Rate	4.167%	3.219%	3.392%
Coeff. of Variation of Wealth	0.0	0.203	0.200
Gini Index of Wealth	0.0	0.108	0.105

by the market alone. As part of the calculations, we obtain a value for  $g$  (the third term in equation (18)) of -0.001147. The condition  $g < 0$  is indeed the condition that ensures less capital in the constrained optimum than in the laissez-faire equilibrium. As a further illustration of the features of this economy, Figure 3 shows the steady-state capital supplies and demands for the laissez-faire as well as constrained-efficient versions of this economy (the full-insurance level of capital is also indicated in the figure: the discontinuous line).

Note that the constrained-efficient supply of steady-state capital is only lower than the market supply of capital for some interest rates—for some interest rates it is higher.

### 5.3 Why is the planner taking opposite actions in these two economies and Robustness

At this point an important question arises, why is the relation between the market and the planner allocations so different in these two model economies? In our view, the best way to explain it is to point out that the utilitarian planner by weighting agents equally in its objective function ends up caring a lot more for helping the agents with a high marginal utility of consumption, that is the consumption poor agents. Recall that the planner cannot transfer resources, all it can do is to bias relative prices in favor of the group of agents that it cares the most. So whether it wants to increase capital relative to the market allocation or decrease it depends on whether by doing so it helps increase the income of the consumption poor agents. Consequently, if consumption-poor agents are labor income intensive relative to consumption rich agents or then the planner wants to increase wages at the expense of

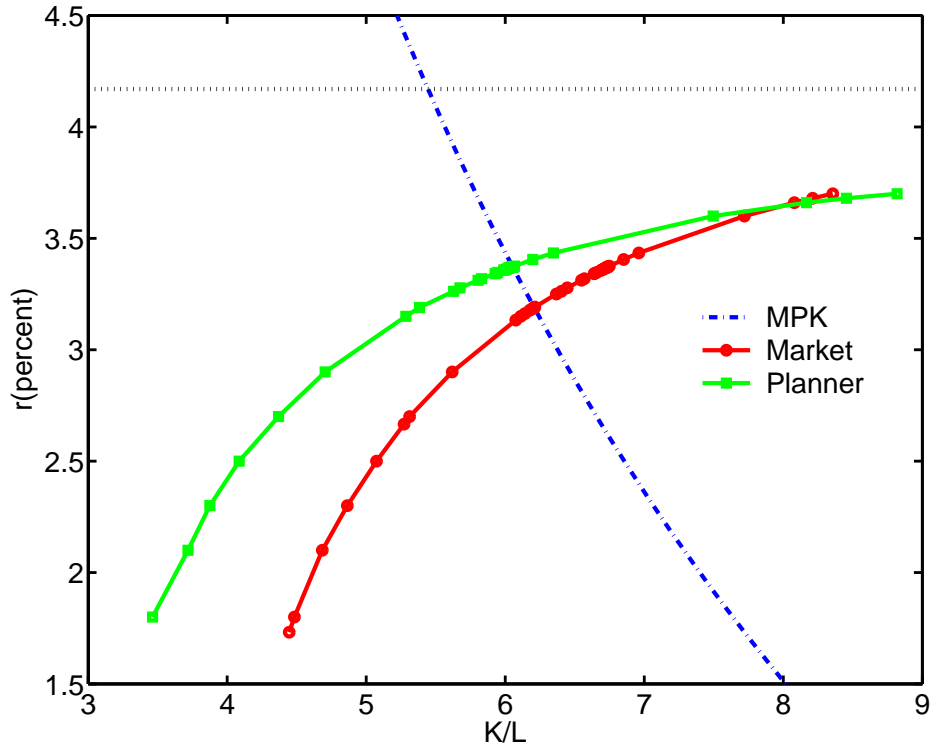


Figure 3: STEADY-STATE CAPITAL SUPPLIES AND DEMANDS FOR THE MARKET AND CONSTRAINED-EFFICIENT UNEMPLOYMENT ECONOMIES

the rates of return.

Figure 4 plots the fraction of income coming from labor for agents sorted by consumption for both the Baseline and the Unemployment economies. We see that for both economies the general pattern is a decreasing relation, consumption rich agents are generally less labor income intensive than the consumption poor. However, in the Unemployment economy there are a few agents, in fact, most of the unemployed that have quite a low share of labor earnings out of total earnings. And it is these agents the ones for which the planner cares the most. It is possible then that those few poorest agents in the Unemployment economy tilt the balance towards the planner choosing to increase the capital stock beyond what the market does as it is indeed the case in our specific calibration.

To further investigate these issues we run some additional economies with different parameterizations. The first panel of Table 5 reports the properties of the market and planner allocations for the Baseline Economy and for two economies where there the persistence of

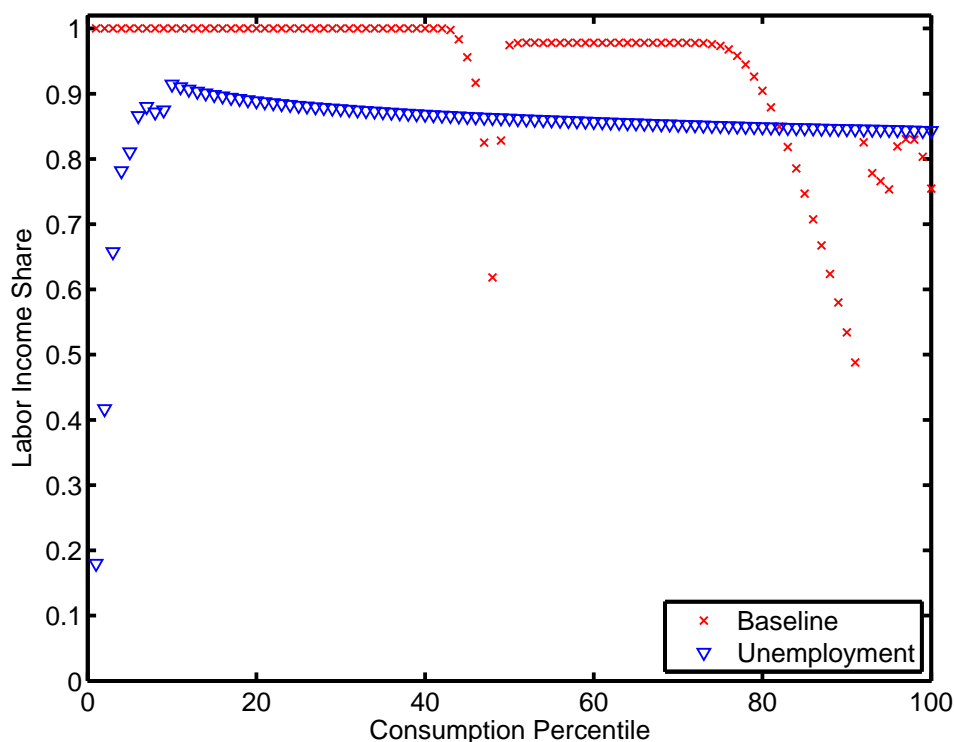


Figure 4: FRACTION OF INCOME COMING FROM LABOR EARNINGS BY CONSUMPTION PERCENTILES

earnings has been reduced. We see that lower persistence both increases wealth in the market economy as lower persistence increases the future risk for wealthier agents and reduces the planner's wealth. The reason for this is that the planner sees the unlucky agents as being less so now because they are more likely to increase their earnings in the future. The quantitative size of these effects is large.

With respect to the changes in the earnings of each type, we see that a 10% increase in the earnings of the poorest reduces total wealth in both the market and the planner allocation, with an overall effect of reducing a tiny bit the relative premium of the planner's allocation. Increasing the middle group's earnings increases a very small amount the market's wealth and a larger amount the market wealth resulting in a larger planner's premium. Finally, increasing the earnings of the highest group increases wealth in both allocations but relatively more the markets.

Changes in the parameterization of the Unemployment Economy also produce nice results

Table 5: **Baseline Economy: Decreasing Persistence and Type-Specific Earnings**

<b>Decreasing Persistence</b>	Deterministic	Baseline ( $\rho = .923$ )		$\rho = .855$		$\rho = .408$	
	Economy	Market	Planner	Market	Planner	Market	Planner
$K$	1.736	4.017	15.668	4.339	12.908	4.396	7.214
$Y$	1.000	1.353	2.208	1.391	2.059	1.397	1.670
$K/Y$	1.736	2.970	7.096	3.120	6.269	3.146	4.320
$r$	12.740%	4.123%	-2.927%	3.539%	-2.257%	3.442%	0.334%
Coef Var	0.0	2.562	2.501	2.544	2.717	1.278	1.461
Wealth Gini	0.0	0.861	0.864	0.836	0.868	0.610	0.655
$(K^p - K^m)/K^m$		2.901		1.975		0.641	

<b>Changing Type-Specific Earnings</b>	10% increase lowest earning		10% increase middle earning		10% increase highest earning	
	Market	Planner	Market	Planner	Market	Planner
	$K$	3.979	15.438	4.071	16.124	4.417
$Y$	1.356	2.209	1.396	2.290	1.443	2.348
$K/Y$	2.935	6.989	2.917	7.040	3.061	7.162
$r$	4.266%	-2.849%	4.340%	-2.886%	3.762%	-2.973%
Coef Var	2.593	2.528	2.481	2.440	2.604	2.552
Wealth Gini	0.868	0.868	0.848	0.856	0.867	0.871
$(K^p - K^m)/K^m$	2.880		2.961		2.807	

that help our intuition. Table 6 displays the effects of changes, first in the duration of unemployment from 2.5 periods to 2.1 and to 1.75 and then in the value of unemployment, from .01 to .1. Both these changes are sufficient to overturn the result that the planner would like less capital than the market. A shorter duration of unemployment would reduce the number of agents that have been unemployed for a while and that therefore have almost drained their savings. A higher value of income while unemployed makes this state less taxing for the unlucky. In these two cases, the left tail of the consumption distribution have a much lower importance, either the number of agents with low consumption is not that low and/or their actual marginal utility of consumption is not that high.

Table 6: **Unemployment Economy**

	$\rho = .6$		$\rho = .5$		$\rho = .4$		higher unemp.	
	(duration 2.5)		(duration 2.1)		(duration 1.75)		benefit ( $b = 0.1$ )	
	Market	Planner	Market	Planner	Market	Planner	Market	Planner
$K$	3.359	3.279	3.224	3.216	3.134	3.173	3.528	3.562
$Y$	1.047	1.038	1.031	1.031	1.021	1.026	1.135	1.139
$K/Y$	3.209	3.160	3.126	3.121	3.070	3.094	3.107	3.126
$r$	3.219%	3.392%	3.516%	3.534%	3.728%	3.635%	3.587%	3.516%
Coef Var	0.203	0.200	0.224	0.224	0.248	0.251	0.246	0.248
Wealth Gini	0.108	0.105	0.124	0.123	0.139	0.141	0.135	0.137
$(K^p - K^m)/K^m$	-0.0236		-0.0024		0.0608		0.0096	

#### 5.4 No constrained-efficient steady state: the original Aiyagari parameterization

We now explore an economy that will turn out not to have a constrained-efficient steady state.<sup>13</sup> This economy is calibrated to the process for earnings considered in Aiyagari (1994), whose parameterization mainly originates from PSID data. We follow Aiyagari very closely; in his model economy, there are very few agents in the right tail the earnings distribution, and the wealth distribution is far from that observed in U.S. data.

We specify the parameters so that the economy with complete markets (the standard neoclassical growth model) satisfies the standard properties. The interest rate is set to 4.167%:  $\beta$  is 0.96). Our only departure from Aiyagari (1994) is to set the intertemporal elasticity of substitution,  $\frac{1}{\sigma}$ , to be equal to 0.5.<sup>14</sup> The capital share is equal to 0.36 and the capital-output ratio is set to slightly under 3 (actually to 2.959 so that the depreciation rate of capital  $\delta$  equals 0.08).

With respect to the process for earnings, Aiyagari (1994) assumes an AR(1) in the logarithm of labor income. The process is fully described by two properties: its persistence and

<sup>13</sup>The discussion in this section presumes that no upper bound is imposed on agents' asset choices. If such a bound is (arbitrarily) imposed, a steady state does exist in which this upper bound binds for a non-negligible subset of agents.

<sup>14</sup>Aiyagari (1994) considers the values 1, 0.33, and 0.2. Ghez and Becker (1975) and MaCurdy (1981), both using a life cycle model and explicitly accounting for leisure, assume much lower values. Mehra and Prescott (1985) and Prescott (1986) discuss other estimates in the literature and conclude that a reasonable number is not too far from 1. Cooley and Prescott (1995) argue that this parameter is very difficult to pin down but also settle for a value of 1. Hurd (1989) has a point estimate below one. We have investigated the properties of the Aiyagari economy for values of the intertemporal elasticity of substitution of 1 and 3 and found the same property of inexistence of a stationary allocation and ever increasing wealth dispersion.

its volatility. Aiyagari (1994) chooses both values following estimates from Kydland (1984), who used PSID data, and from Abowd and Card (1987) and Abowd and Card (1989), who used both PSID and NLS data. Then, Aiyagari approximates this process using a seven-state Markov chain following the procedures described in Tauchen (1986). We follow the same procedure, although we reduce the Markov chain to three states. We take our benchmark to have an autocorrelation of 0.6 and a coefficient of variation of 0.2.<sup>15</sup> In Table 7, we report the parameter values of the Aiyagari (1994) model economy.<sup>16</sup> The steady state of the market

Table 7: **Parameter values of the Aiyagari (1994) model economy**

General Parameters	$\beta$	$\sigma$	$\theta$	$\delta$
	0.96	2	0.36	0.08
Earnings	$e \in \{e_1, e_2, e_3\} =$	{.78,	1.00,	1.27}
	$\pi_{e,e'} =$	$\begin{bmatrix} 0.66 & 0.27 & 0.07 \\ 0.28 & 0.44 & 0.28 \\ 0.07 & 0.27 & 0.66 \end{bmatrix}$		
Stat. Distribution	$\pi^* =$	0.337	0.326	0.337

economy has 2.03% more assets and 0.70% more assets than its full-insurance counterpart, with an interest rate of 4.011% instead of the 4.167%. The implied coefficient of variation of wealth is 0.718 and its Gini Index of wealth is 0.388, far below the dispersion observed in data.<sup>17</sup>

The interesting feature of this economy, however, is that the planner's problem does not have a steady state. To understand this result, we first display the aggregate steady-state capital demands and supplies for various interest rates for the market and the constrained-efficient economies: see Figure 5.

A steady state requires the intersection of the demand curve with the supply curve of the planner. But the latter is not defined beyond a certain interest rate, which is around 2.15%

<sup>15</sup>Aiyagari (1994) provides results for autocorrelations of 0, 0.3, 0.6, and 0.9 and for coefficients of variation of 0.2 and 0.4.

<sup>16</sup>We have also ran other experiments where we use the values for the IES used in Aiyagari (1994) and we obtain the same results.

<sup>17</sup>For a discussion, see Castañeda, Díaz-Giménez, and Ríos-Rull (2003).

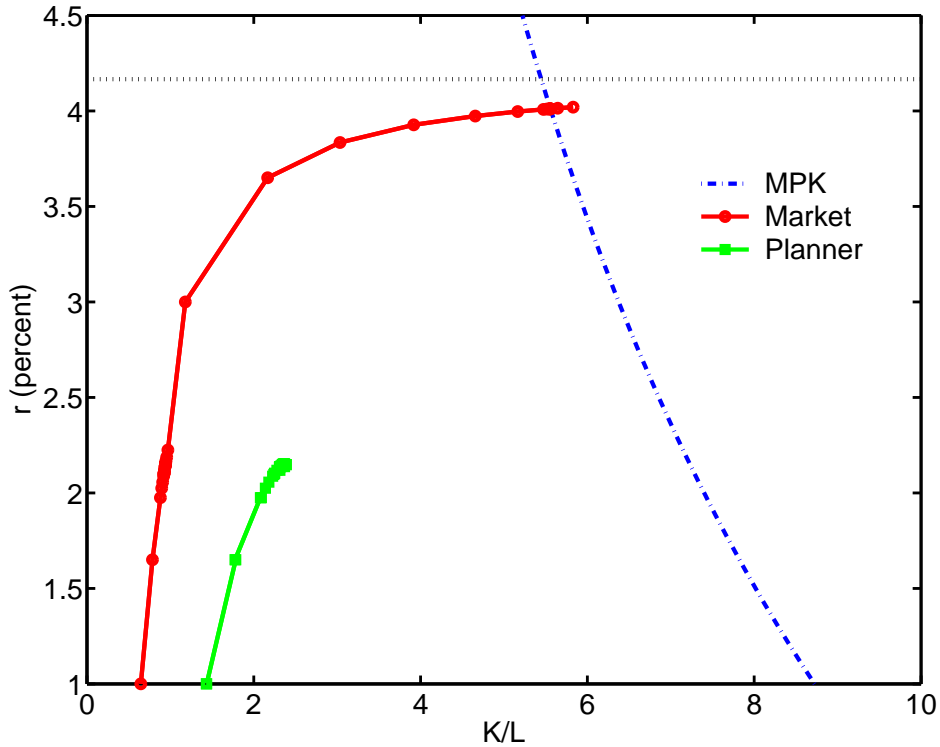


Figure 5: STEADY-STATE CAPITAL SUPPLIES AND DEMANDS FOR THE AIYAGARI (1994) ECONOMY.

in the picture. Thus no intersection exists.

To understand this feature, it is convenient to first look more closely at the individual savings functions of market economies, as indexed by different constant interest rates. These are shown in Figure 6.

Since  $g = 0$  underlies these savings functions—the market does not take into account the additional term that is in the first-order condition of a constrained-efficient allocation. As the interest rate rises, so does the intersection between the savings function and the diagonal (45-degree line). As the interest rate approaches the inverse of the discount rate the intersection goes to infinity, but for any interest rate below this value there is a finite intersection (see Huggett (1997) for a formal verification of this feature).

When  $g > 0$ , which would be the typical feature of a constrained-efficient steady state, the constrained-efficient savings function looks quite different. This is illustrated in Figure 7.



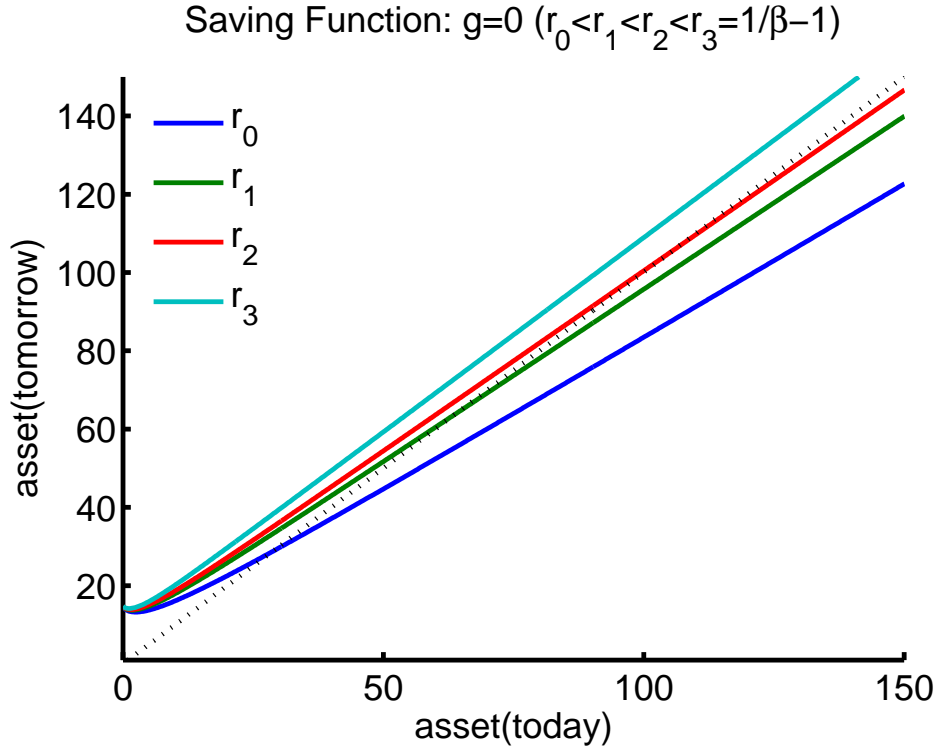


Figure 6: MARKET SAVINGS FUNCTIONS FOR DIFFERENT VALUES OF  $r$ .

For a given interest rate, this figure plots the savings rule for various levels of  $g$ . The form of the saving function is convex and it shifts upward as  $g$  increases. This implies that there is a maximum level of  $g$  that is consistent with the existence of an upper bound of assets holdings. For values of  $g$  above this level, such as  $g_3$ , there is no such upper bound, implying that aggregate assets must increase without bound, preventing the existence of a steady state for the constrained-efficient economy.

We now characterize the solution, which involves a nonstationary path for capital.

#### 5.4.1 The solution to the planner's problem

To solve the planning problem numerically, we need to characterize a whole time path for the wealth distribution. For this, we use similar techniques to those developed in Krusell and Smith (1997) and Krusell and Smith (1998); for details, see the appendix. For simplicity, we consider an initial condition not of complete equality, but rather the steady-state distribution of the market economy. Figure 8 shows the properties of the solution for the initial condition

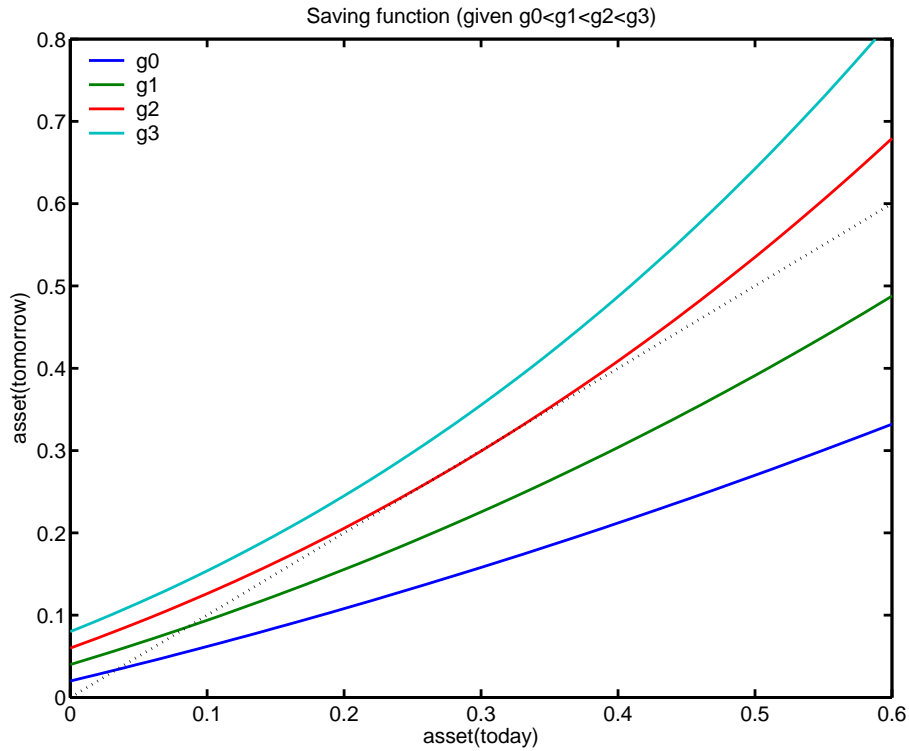


Figure 7: PLANNER’S SAVINGS FUNCTION FOR  $e^3$  FOR DIFFERENT VALUES OF  $g$ . FOR  $g > g_2$  THERE IS NO STEADY STATE.

given by the market steady state.

We see increasing capital accumulation toward a steady-state value for capital. This reveals a path with  $g_t > 0$ .

The lack of convergence here is manifested through a distribution of capital that becomes more and more concentrated over time, as illustrated in Figure 9, which shows the measure of people with zero asset holdings over time.

This measure becomes larger and larger, while the average asset holding stays bounded around 10. Thus, a more and more vanishingly small fraction of the agents hold the bulk of the capital over time, leading to a limit where a measure-zero set of agents holds all the capital.

The origin of the extreme long-run wealth distribution is a form of “increasing returns” to saving of the constrained optimum, as illustrated above in the convex savings rules. The third

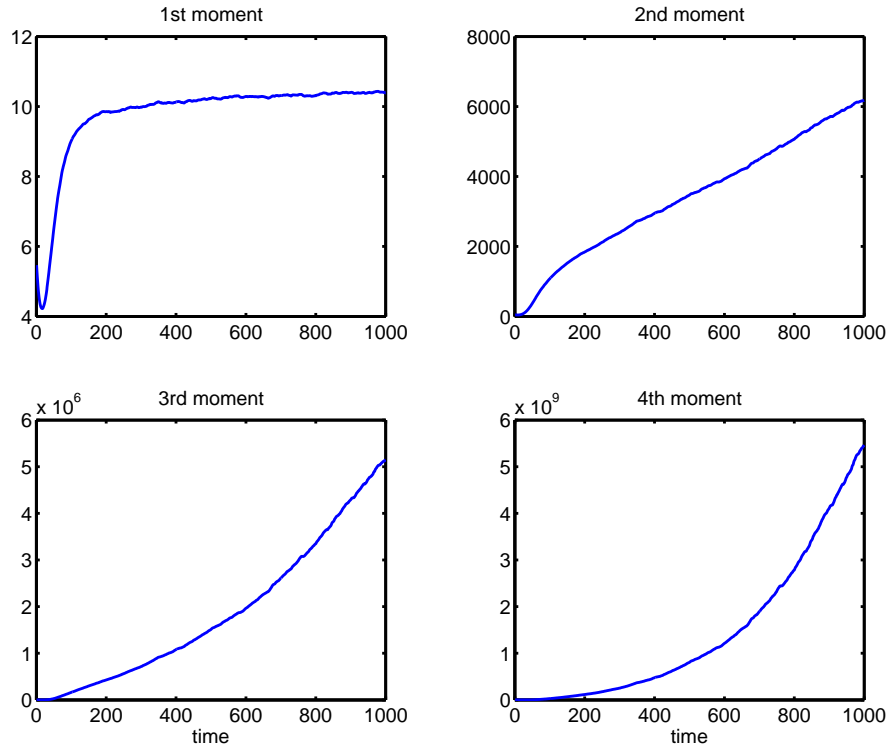


Figure 8: TRANSITION FROM THE MARKET STEADY STATE TO THE CONSTRAINED OPTIMUM

term in the first-order condition, the constant  $g$ , which is an additional marginal benefit from savings, is particularly large in relative terms for rich consumers, whose marginal utilities of consumption are low: the more you save, the more important is this term for further savings decisions.

## 6 Conclusion

In this paper we have investigated the welfare properties of the one-sector neoclassic growth model with uninsurable idiosyncratic shocks. We have relied on the constrained efficiency concept used in the general equilibrium literature, and we have demonstrated constrained inefficiency for our model. We have also provided a characterization of constrained efficiency by means of the first-order condition of a planner's problem that points to the margins of relevance for whether capital is too high or too low in equilibrium: the income composition of the (consumption-)poor. The mechanism is simple: if the income of the poor comes mainly from labor earnings, ex-ante (constrained-efficient) risk considerations demand that

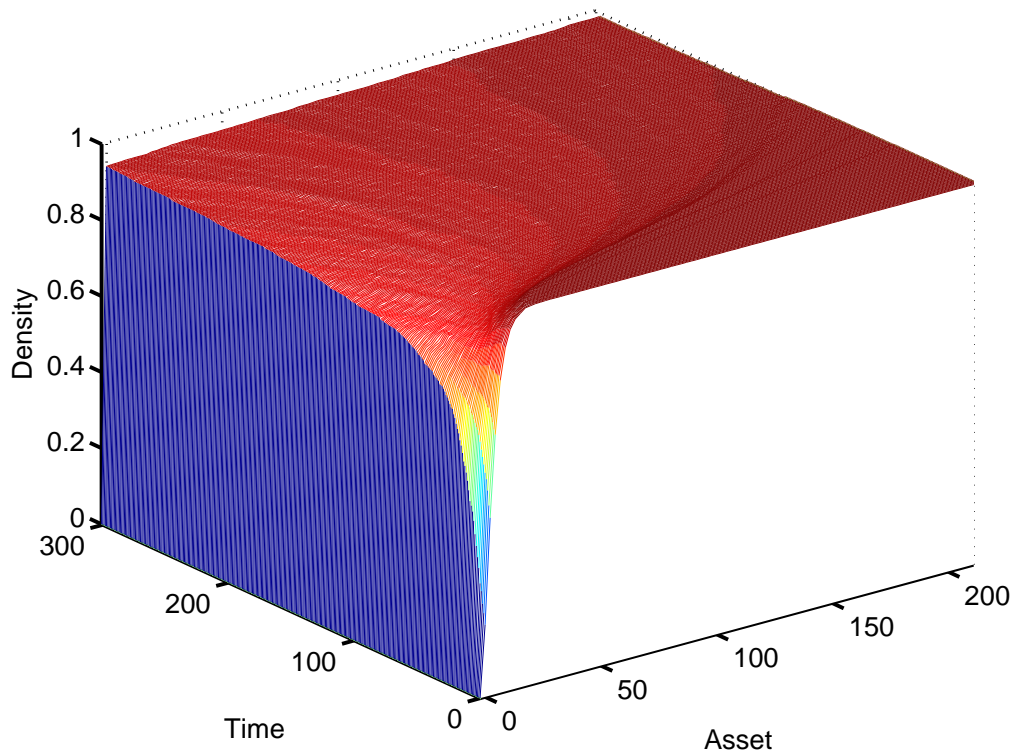


Figure 9: DENSITY FUNCTION FOR ASSETS FOR EACH PERIOD: AIYAGARI MODEL

the long-run capital stock be increased, because that raises wages and therefore the income of the poor. We then calibrated our benchmark model parameters governing idiosyncratic risks to the U.S. earnings and wealth distribution, and for this distribution the income of the poor is mainly composed of labor earnings. Using numerical model solution, we then computed the constrained-efficient steady state for our model economy, and we concluded that the long-run capital stock in a laissez-faire world is not only too low, but much too low.

We also illustrated how the income composition of the poor matters by looking at alternative parameterizations, which indeed yield qualitatively different results than we obtained for our benchmark economy. For a parameterization where risk is unemployment risk (thus not properly depicting the actual earnings distribution), the laissez-faire capital stock is too high, because now the poor have no labor income. For the parameterization of earnings in Aiyagari (1994) (which generates too little dispersion in earnings and in wealth), we find that, as in our benchmark economy, laissez-faire capital is too low. In this case, however, the constrained-efficient outcome involves ever-increasing inequality, something we did not find

for our benchmark economy. Ever-increasing inequality can be understood from the perspective of the planner's first-order condition, which reveals that there is increasing returns of sorts to constrained-efficient individual savings.

Several questions remain open. One concerns the calibration of individual risk. We follow Castañeda, Díaz-Giménez, and Ríos-Rull (2003) here, but there are alternative ways of explaining the wide dispersion in wealth that we observe in most economies. One relies on preference heterogeneity, for example as pursued in Krusell and Smith (1997) and Krusell and Smith (1998). There, rich consumers are rich to a large extent because they, or their forefathers, were particularly patient and saved; conversely, the poor are poor to a large extent because they chose to be poor—they chose to consume in the past. The specific modeling there views discount factors as random (and uninsurable), and thus people are all the same *ex ante*, but with some persistence in discount factors, significant wealth inequality can be generated even from small cross-sectional heterogeneity. From the perspective of the present paper, a model where wealth inequality derives chiefly from persistent shocks to patience might give quite different results than those we obtain here, since they suggest that wealth inequality is not all a result of incomplete risk sharing. In other words, the planner would be more willing to let those who choose to become poor (rich) stay poor (rich). Similarly, other forms of heterogeneity in preferences (such as risk aversion) or in individuals' abilities or opportunities (e.g., possibly making it harder for some to participate in asset markets than it is for others) would also be valuable to examine from the perspective of constrained efficiency. Ultimately, microeconomic studies hopefully allow us to better distinguish which elements of individual heterogeneity are key and which are not. The present paper takes a somewhat conservative position by not allowing any other heterogeneity than in earnings, but we firmly believe that it is a useful one and a very natural first step. From the perspective of what causes inequality, it is perhaps not conservative enough: it is perhaps too "left-wing", because it interprets all successes as luck and all failures as bad luck.

Compared to the findings in Aiyagari (1995) and Aiyagari and McGrattan (1995), the conflicting policy implications are apparent. The question is: should capital taxes be positive or negative (raised or lowered)? A precautionary-savings perspective suggests that there is overaccumulation of capital in the *laissez-faire* economy, and that capital taxation can improve on the allocation. Such a conclusion is a noteworthy one, since it does alter otherwise standard prescriptions (see, e.g., Chamley (1986)) that call for zero long-run capital taxes.

Our findings here, based on the same precautionary-savings model but with a different notion of efficiency, indicate that a move back toward zero taxes may be beneficial, and they actually suggest that a “radical” move back may be called for: the constrained optimum in our economy has a higher capital stock than that of the first-best allocation. Of course, this indication is based on taking a different stand on what is feasible for the planner; we allow some things that are not allowed in Aiyagari (1995) and Aiyagari and McGrattan (1995), and vice versa. A natural next step of the present analysis is to consider more restrictions on how the planner can alter outcomes. For example, it would be interesting to explore nonlinear taxes on wealth that are not allowed to be state-contingent.

It is also instructive to compare our findings to those in Kehoe and Levine (1993), who consider a complete-markets (endowment) economy with limited commitment/constraints on enforcement. They prove, for their economy, that the competitive equilibrium is “conditionally efficient”, which in a world with one good coincides with the constrained-efficiency concept from the incomplete-markets literature that we employ here. One could interpret the lack of contingent-claims markets in our two-period economy as simply reflecting a lack of enforcement: agents can save but any additional contracts contingent on idiosyncratic risk would necessarily involve “paying back” in some states of nature, and therefore such contracts are not feasible. The fact that we find constrained inefficiency here should not be surprising, however, because we do have more than one good per period, (capital services, labor, and consumption), and relative price changes make the notion of conditional efficiency not apply. Our economy is thus only one of many examples where conditional efficiency does not imply constrained efficiency.

# Appendix

## A Proofs

**Proposition 1.** *If the distribution  $x$  admits a density, the first-order condition of the solution to problem (14) is, for all  $(a, e) \in S$ ,*

$$\begin{aligned}
& u_c(a [1 + r(x)] + e w(x) - h^*(x, e, a)) \geq \\
& \beta [1 + r'(x')] \sum_{e'} \pi_{ee'} u_c(h^*(x, e, a) [1 + r'(x')] + e' w'(x') - a'') \\
& + \beta \sum_{e'} \int_S \pi_{\tilde{e}e'} u_c(h^*(x, \tilde{e}, \tilde{a}) [1 + r'(x')] + e' w'(x') - a'') \\
& [h^*(x, \tilde{a}, \tilde{a}) f_{KK}(K'(x')) + e' f_{LK}(K'(x'))] dx(\tilde{a}, \tilde{e}) \quad (19)
\end{aligned}$$

where we use  $a''$  as an abbreviation for  $h^*(x', e', h^*(x, e, a))$ , and where  $x' = Y(x) \equiv T(x, y)$ , and again the inequality becomes equality if  $h^*(x, e, a) > \underline{a}$ .

*Proof.* The sequence of policy rules  $h_t$  by which, given a distribution of savings and labor at period  $t$  with density  $x_t$ , a planner would instruct an agent endowed with labor  $e$  and capital  $a$  to save  $h_t(x_t, e, a)$  must solve

$$\begin{aligned}
& \max_{h_t} \sum_t \beta^{t-1} \sum_e \int u(c_t) x_t(a, e) da \\
& \text{s.t. } c_t + h_t(x_t, e, a) = a[1 + f_K(K(x_t))] + e f_L(K(x_t))
\end{aligned}$$

given  $x_1$ , where

$$K(x_t) = \sum_e \int a x_t(a, e) da$$

is the aggregate capital at period  $t$ . The updating operator for the sequence of distribution densities  $x_t$  of savings  $a$  and labor  $e$  is

$$x'(a', e') = \sum_e \pi_{ee'} \frac{x((h^*)^{-1}(x, e, a'), e)}{\frac{d}{da} h^*(x, e, (h^*)^{-1}(x, e, a'))}$$

at any period  $t$ .<sup>18</sup> Therefore, the planner's optimal policy rule  $h^*$  that instructs an agent

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<sup>18</sup>Here  $(h^*)^{-1}(x, e, a')$  denotes value at  $a'$  of the inverse of  $h^*(x, e, \cdot)$ , for given  $e$  and  $x$ . In effect, note that assuming the policy rule  $h^*(x, e, \cdot)$  is increasing for all  $e$ , then the mass of agents endowed with labor  $e'$  and capital  $\tilde{a}'$  in any given interval  $(0, a')$  next period is

$$\int_0^{a'} x'(\tilde{a}', e') d\tilde{a}' = \sum_e \pi_{ee'} \int_0^{(h^*)^{-1}(x, e, a')} x(a, e) da$$

with  $e$  and  $a$  today to save the amount  $h^*(x, e, a)$ , given a distribution of labor and savings today with density  $x$  and the savings  $a''$  tomorrow, must maximize

$$\begin{aligned} & \sum_e \int u(a[1 + f_K(K(x))] + ef_L(K(x)) - a')x(a, e)da \\ & + \beta \sum_{e'} \int u(a'[1 + f_K(K(x'))] + e'f_L(K(x')) - a'')x'(a', e')da', \end{aligned}$$

with

$$x'(a', e') = \sum_e \pi_{ee'} \frac{x((h^*)^{-1}(x, e, a'))}{\frac{d}{da}h^*(x, e, (h^*)^{-1}(a', e, x))},$$

with respect to  $a'$ . Merging the sums over  $e$  and the integration with respect to  $a$ , rearranging terms, and making the necessary changes of variables,  $h^*$  should thus maximize

$$\begin{aligned} & \sum_e \int \left[ u(a[1 + f_K(K(x))] + ef_L(K(x)) - h^*(x, e, a)) \right. \\ & \left. + \beta \sum_{e'} \pi_{ee'} u(h^*(x, e, a)[1 + f_K(K'(x'))] + e'f_L(K'(x')) - a'') \right] \cdot x(a, e)da, \end{aligned}$$

where, using the same changes of variables,

$$K'(x') = \sum_{e'} \int a' \sum_e \pi_{ee'} \frac{x((h^*)^{-1}(x, e, a'), e)}{\frac{d}{da}h^*(x, e, (h^*)^{-1}(a', e, x))} da' = \sum_e \int h^*(x, e, a)x(a, e) da.$$

For any variation  $\delta_{h_{e_0}^*}$  of the optimal policy rule  $h^*(x, e_0, \cdot)$  that determines the savings of the agents endowed with a given level of labor  $e_0$  and given a distribution  $x$ , and for any  $\varepsilon \neq 0$ , the policy rule

$$h^\varepsilon(x, e, a) = h^*(x, e, a) + \varepsilon \chi_{e=e_0} \delta_{h_{e_0}^*}(a),$$

where  $\chi_{e=e_0}$  is the indicator function for  $e = e_0$ , should be suboptimal. Therefore, the derivative with respect to  $\varepsilon$  at 0 of

$$\begin{aligned} \psi(\varepsilon) &= \sum_e \int \left[ u(a[1 + f_K(K(x))] + ef_L(K(x)) - y^\varepsilon(a, e, x)) \right. \\ & \left. + \beta \sum_{e'} \pi_{ee'} u(h^\varepsilon(x, e, a)[1 + f_K(K'(T(x, h^\varepsilon)))] + e'f_L(K'(T(x, h^\varepsilon))) - a'') \right] \cdot x(a, e)da \end{aligned}$$

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from which the updating operator follows taking the derivative on both sides with respect to  $a'$ , the upper limit of the integral on the left-hand side.



must be 0. That is to say,

$$\begin{aligned}
\frac{d}{d\varepsilon}\psi(0) &= \int \left[ -u_c(a[1 + f_K(K(x))] + e_0 f_L(K(x)) - h^*(x, e_0, a)) \delta_{h_{e_0}^*}(a) \right. \\
&\quad + \beta \sum_{e'} \pi_{e_0 e'} u_c(h^*(x, e_0, a)[1 + f_K(K'(x'))] + e' f_L(K'(x')) - a'') \\
&\quad \quad \cdot \delta_{h_{e_0}^*}(a)[1 + f_K(K'(x'))] \left. \right] x(a, e_0) da \\
&\quad + \beta \sum_{ee'} \pi_{ee'} \int \left[ u_c(h^*(x, e, a)[1 + f_K(K'(x'))] + e' f_L(K'(x')) - a'') \right. \\
&\quad \quad \cdot [h^*(x, e, a) f_{KK}(K'(x')) + e' f_{LK}(K'(x'))] \left. \int \delta_{h_{e_0}^*}(\tilde{a}) x(\tilde{a}, e_0) d\tilde{a} \right] x(a, e) da = 0.
\end{aligned}$$

Since this must hold for any variation, it must hold in particular for  $\chi_{a \geq a_0}$ , the indicator function of  $a \geq a_0$ , for any  $a_0$ . Therefore, rearranging terms and substituting  $\chi_{a \geq a_0}$  for  $\delta_{h_{e_0}^*}$ ,

$$\begin{aligned}
&\int_{a_0}^{+\infty} \left[ -u_c(a[1 + f_K(K(x))] + e_0 f_L(K(x)) - h^*(x, e_0, a)) \right. \\
&\quad + \beta \sum_{e'} \pi_{e_0 e'} u_c(h^*(x, e_0, a)[1 + f_K(K'(x'))] + e' f_L(K'(x')) - a'') \\
&\quad \quad \cdot [1 + f_K(K'(x'))] \left. \right] x(a, e_0) da \\
&\quad + \beta \sum_{ee'} \pi_{ee'} \int \left[ u_c(h^*(x, e, a)[1 + f_K(K'(x'))] + e' f_L(K'(x')) - a'') \right. \\
&\quad \quad \cdot [h^*(x, e, a) f_{KK}(K'(x')) + e' f_{LK}(K'(x'))] \left. \int_{a_0}^{+\infty} x(\tilde{a}, e_0) d\tilde{a} \right] x(a, e) da = 0.
\end{aligned}$$

The right-hand side of the last expression is therefore a constant function of  $a_0$  equal to 0. As a consequence, its derivative with respect to  $a_0$  must be 0, i.e., for all  $a_0$  and all  $e_0$ ,

$$\begin{aligned}
&-u_c(a_0[1 + f_K(K(x))] + e_0 f_L(K(x)) - h^*(x, e_0, a_0)) \\
&\quad + \beta \sum_{e'} \pi_{e_0 e'} u_c(h^*(x, e_0, a_0)[1 + f_K(K'(x'))] + e' f_L(K'(x')) - a'') \cdot [1 + f_K(K'(x'))] \\
&\quad + \beta \sum_{ee'} \pi_{ee'} \int u_c(h^*(x, e, a) f_K(K'(x')) + e' f_L(K'(x')) - a'') \\
&\quad \quad \cdot [h^*(x, e, a) f_{KK}(K'(x')) + e' f_{LK}(K'(x'))] x(a, e) da = 0,
\end{aligned}$$

i.e., for all  $(a, e)$

$$\begin{aligned}
& u_c(a[1+r(x)] + ew(x) - h^*(x, e, a)) = \\
& \beta[1+r'(x')] \sum_{e'} \pi_{ee'} u_c(h^*(x, e, a)[1+r'(x')] + e'w'(x') - a'') \\
& + \beta \sum_{\tilde{e}e'} \pi_{\tilde{e}e'} \int u_c(h^*(x, \tilde{e}, \tilde{a})[1+r'(x')] + e'w'(x') - a'') \\
& \quad \cdot [h^*(x, \tilde{e}, \tilde{a})f_{KK}(K'(x')) + e'f_{LK}(K'(x'))]x(\tilde{a}, \tilde{e})d\tilde{a}.
\end{aligned}$$

Q.E.D. □

## B Approximation Method

We now describe briefly how we solve the problem of the planner outside the steady state.

Given  $V(m)$  for  $m \in \mathbb{R}^{nm}$  and  $x$ , we can solve the following problem

$$\max_{y(e,a)} \int u(a[1+r(K)] + e w(K) - y(e, a)) dx + \beta V(m') \quad (20)$$

$$\text{subject to} \quad m' = \varphi(x, y). \quad (21)$$

We select a set of moments  $m \in \mathbb{R}^{nm}$  of the distribution  $x$ ; the moments are defined by  $m^i = \int_S a^i dx$ , and typically we simply choose the first three moments. We then approximate the function  $V$  as a quadratic function of the logarithms of the chosen moments of the distribution, with some coefficient restrictions. A typical functional form that we have chosen is

$$V(m) = \alpha_0 + \alpha_1 \log(m^1) + \alpha_2 \log(m^2) + \alpha_3 \log(m^3) + \alpha_4 \log^2(m^1).$$

To solve the planning problem and to obtain the coefficients  $\alpha$  we proceed as follows.

1. Make an initial guess on  $V$ , labeled  $V^0$ . That is, guess an  $\alpha^0 \in \mathbb{R}^{n\alpha}$ .
2. Choose an initial distribution  $x_0$  and calculate its moments  $m_0$ . Then generate a sample for 10,000 individuals.

3. Construct a time series of distributions,  $x_{t+1}$ , decision rules,  $y_t$ , moments,  $m_{t+1}$ , and current returns,  $R_t$ , with the following iterative procedure for  $T$  periods (we choose  $T = 3,000$ ). Given  $x_t$  and its moments  $m_t$ ,

(a) solve problem (20) to obtain decision rules  $y_t$ :

i. choose the grid points for asset holdings (here it is important to assign more points on the lower asset range to better approximate decisions of agents with lower asset holdings),

ii. with problem (20) taking the form

$$\max_{y_t(e,a)} \int_{S_t} u(a[1+r_t(m_t^1)] + e w_t(m_t^1) - y_t(e,a)) dx_t + \beta V^0(m_{t+1})$$

subject to 
$$m_{t+1}^i = \int_{S_t} y_t^i dx_t,$$

iii. and find  $y_t(e,a)$  satisfying the associated first-order condition for all  $\{e,a\} \in S$ , i.e.,

$$u_c(a[1+r_t(m_t^1)] + e w_t(m_t^1) - y_t(e,a)) \geq \beta \sum_{i=1}^{n_m} V_i^0(m_{t+1}) i [y_t(e,a)]^{i-1};$$

(b) update the distribution and calculate its moments  $x_{t+1}$  and  $m_{t+1}$ :

$$\begin{aligned} x_{t+1} &= T(x_t, Q(\cdot, y_t)) \\ m_{t+1}^i &= \int_{S_t} y_t^i dx_t; \end{aligned}$$

(c) and, finally, use the obtained distribution and decision to compute current returns  $R_t$  using

$$R_t = \int_{S_t} u[a(1+r(x_t)) + e w(x_t) - y_t(e,a)] dx_t,$$

going back to 3a until  $t = T$ .

4. Use current returns and moments as well as  $V^0$  to calculate a new set of values for all  $t$ , using

$$V_t' \equiv R_t + \beta V^0(m_t)$$

5. Define  $V^1(m)$  by running a regression of the set of values  $\{V'_t\}$  on the moments  $\{m_t\}$  to obtain the new coefficients  $\alpha^1$ .
6. Compare  $V^0$  with  $V^1$ . If these functions are not sufficiently similar, update  $V^0$  using  $V^1$  and go back to 2; otherwise, stop.<sup>19</sup>

We follow this procedure for various combinations of first three moments of  $x$ . All of these imply ever-increasing inequality, with the first moment remaining stationary. Table 8 shows time series data for aggregate assets and for the coefficient of variation of assets over time, for various specifications of function  $V$ . The top panel illustrates that the first moment settles down and becomes stationary after a few hundred periods, whereas the bottom panel shows that inequality keeps increasing over time.

We also applied this procedure to the economies that admit steady states (e.g., our benchmark economy) and here we do find convergence to the steady states of the planner economy: all the four moments settle down relatively quickly, starting from an initial condition given by the steady state of the laissez-faire economy.

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<sup>19</sup>The accuracy of the obtained solution can be judged by the errors in the regression, once a fixed point in coefficients is found.

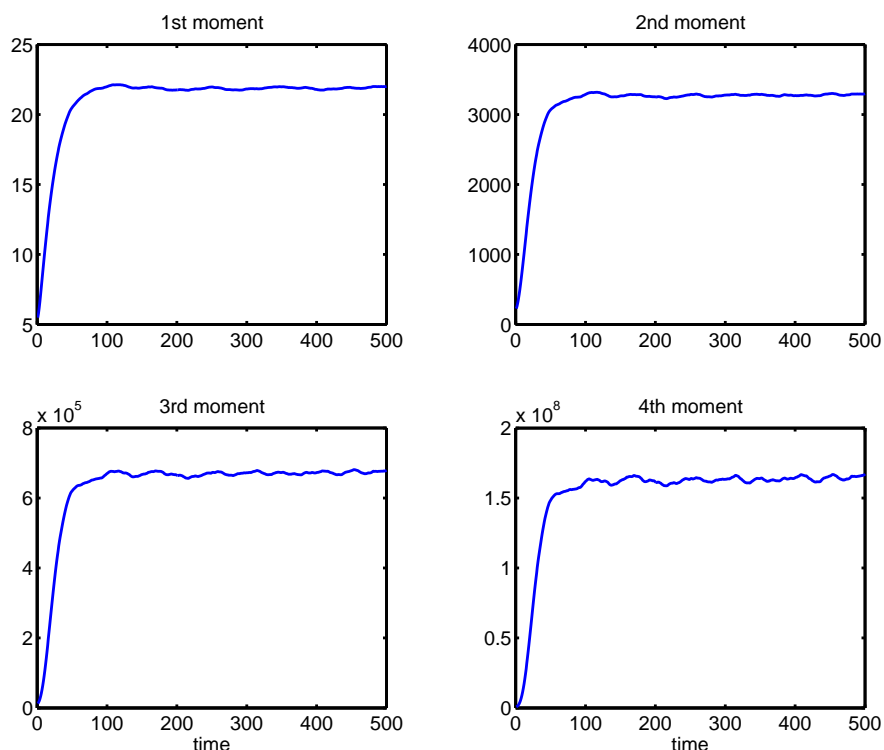


Figure 10: TRANSITION FROM THE MARKET-ECONOMY STEADY STATE TO THE CONSTRAINED OPTIMUM: BENCHMARK MODEL

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Table 8: **Time Series: Simulated Data for the Aggregate Economy**

time	0	200	400	600	800	1000	1200	1400	1600	1800	2000
<u>Aggregate Asset</u>											
<b>A</b>	5.59	9.59	9.97	10.08	10.13	10.18	10.24	10.27	10.30	10.31	10.28
<b>B</b>	5.59	9.71	10.06	10.16	10.22	10.26	10.31	10.34	10.37	10.39	10.35
<b>C</b>	5.59	9.07	9.72	9.95	10.04	10.11	10.20	10.25	10.29	10.32	10.30
<b>D</b>	5.59	6.77	9.28	9.76	9.97	10.10	10.21	10.26	10.31	10.33	10.31
<u>Coeff. of Variation</u>											
<b>A</b>	0.72	4.12	5.10	5.74	6.33	6.88	7.36	7.87	8.37	8.87	9.39
<b>B</b>	0.72	4.86	6.26	7.37	8.35	9.25	10.13	10.96	11.70	12.35	12.93
<b>C</b>	0.72	3.27	4.39	5.20	5.91	6.66	7.40	8.12	8.77	9.31	9.84
<b>D</b>	0.72	2.10	3.75	4.85	5.79	6.70	7.51	8.19	8.81	9.39	10.00

Specifications for  $V$ :

$$\mathbf{A} : \alpha_0 + \alpha_1 \log(m^1) + \alpha_2 \log(m^2) + \alpha_3 \log(m^3) + \alpha_4 \log^2(m^1)$$

$$\mathbf{B} : \alpha_0 + \alpha_1 \log(m^1) + \alpha_2 \log(m^2) + \alpha_3 \log^2(m^1)$$

$$\mathbf{C} : \alpha_0 + \alpha_1 \log(m^1) + \alpha_2 \log(m^2) + \alpha_3 \log^2(m^1) + \alpha_4 \log^2(m^2)$$

$$\mathbf{D} : \alpha_0 + \alpha_1 \log(m^1) + \alpha_2 \log(m^2) + \alpha_3 \log^2(m^1) + \alpha_4 \log(m^1) \log(m^2)$$