Temporal Aggregation Bias and Mixed Frequency Estimation of a New Keynesian model (Job Market Paper)

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Abstract

This paper asks whether frequency misspecification of a New Keynesian model results in temporal aggregation bias of the Calvo parameter. First, when a New Keynesian model is estimated at a quarterly frequency while the true data generating process is the same but at a monthly frequency, the Calvo parameter is upward biased and hence implies longer average price duration. This suggests estimating a New Keynesian model at a monthly frequency may yield different results. However, due to mixed frequency datasets in macro time series recorded at quarterly and monthly intervals, an estimation methodology is not straightforward. To accommodate mixed frequency datasets, this paper proposes a data augmentation method borrowed from Bayesian estimation literature by extending MCMC algorithm with "Rao-Blackwellization" of the posterior density. Compared to two alternative estimation methods in context of Bayesian estimation of DSGE models, this augmentation method delivers lower root mean squared errors for parameters of interest in New Keynesian model. Lastly, a medium scale New Keynesian model is brought to the actual data, and the benchmark estimation, i.e. the data augmentation method, finds that the average price duration implied by the monthly model is 5 months while that by the quarterly model is 20.7 months.

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1 Introduction

The key feature of New Keynesian models is Calvo type of price friction which is central to understand the propagation mechanism of real activities in response to monetary policy shock. The degree of this price friction is parameterized by a probability in which a firm cannot reoptimize within one period. Due to its quantitative importance, Calvo parameter has been extensively studied from both micro-dataset and macro models but its estimates are widely dispersed. From micro dataset, Bils & Klenow (2003) finds the median price duration of firms to be 4.3 months while Nakamura & Steisson (2008) argues 8.7 months once irrelevant pricing behaviors such as temporary salescuts are controlled. On the other hand, the average price duration implied by Calvo parameter in macro models ranges from 8 months to 24 months depending on the models and estimation strategies[1].

This discrepancy between microevidence and macro model's estimates is generally perceived as a misspecification of New Keynesian models. But this perception is neglecting the fact that microevidence is based on monthly observations while macro models are estimated at a quarterly frequency. If a temporal aggregation bias in Calvo parameter is present, modeling the pricing behavior at a coarse frequency while the true decision time interval is shorter can in fact be misleading. Thus, this paper asks whether a frequency misspecification of a New Keynesian model results in a temporal aggregation bias of the Calvo parameter. When a New Keynesian model is estimated at a quarterly frequency while the true data generating process is the same model but at a monthly frequency, the Calvo parameter is upward biased and hence implies longer average price duration. This suggests estimating a New Keynesian model at a monthly frequency may yield different results.

In order to resolve the temporal aggregation bias caused by the frequency misspecification in DSGE models, estimation of a model at the true frequency is necessary. However, when a monthly specified DSGE model brought to an estimation, a technical challenge emerges since data is available at different frequencies. For example, interest rate, inflation rate, wage rate and consumption are available at monthly frequency while GDP and investment are only at quarterly. One way is to identify the analytical mapping from a monthly specification of a model to a quarterly specification. In this way, the converted model at the quarterly frequency can be estimated with quarterly data. This conversion

[1] Christiano, Eichenbaum & Evans (2005), Smets & Wouters(2007), and Del Negro & Schorfeheide (2008) are only a few of many other examples.

is well known with simple statistical models, for example, monthly AR(1) is equivalent to quarterly ARMA(1,1)[2]. However, it is not clear how to convert a monthly specified DSGE model to a quarterly specification in general due to the forward looking nature of decision variables. Therefore, a standard estimation procedure needs to be modified to estimate monthly specified DSGE models.

To accommodate mixed frequency dataset, the quarterly data can be treated as an observable variable that has missing observations. Then, "imputing some values" to these missing observations makes a complete dataset that facilitates the standard procedure of estimation. This is called a data augmentation and this paper proposes an estimation strategy using this data augmentation technique that is borrowed from Bayesian estimation literature in which "imputing some values" to missing observations is based on simulations. The simulation of missing observations is by a direct sampling from a distribution of missing observations. This distribution in general can be expressed by a marginal distribution of a joint distribution that is defined not only in terms of model's parameters but also jointly in terms of an auxiliary variable for missing observations conditional on the available data. Accordingly, MCMC algorithm can be extended to sample both parameters and missing observations similar to sampling a mixture model. MCMC theories have proven this modified algorithm converges at the geometric rate and thus central limit theorem ensures the consistency of marginal sampled estimates for parameters[3]. Gelfand & Smith(1990) refers this joint distribution as "Rao-Blackwellization" of an original target distribution of parameters since this is a form of a generalization of the distribution and thus incorporates richer information by allowing "imputing values" into missing observations which is in some sense an example of Rao-Blackwell Theorem. And they show theoretically that the advantage of the data augmentation is efficiency gains of parameters' estimates and this was further generalized by Liu, Wong & Kong(1994).

A sampling scheme for the data augmentation procedure is not unique at least in the context of estimating DSGE models since missing observations are multiple periods. However, due to the general structure of DSGE models, it is not feasible to sample the whole missing observations in one step which would have been the most efficient method. Instead, this paper chooses to sample missing observations sequentially period by period using information from adjacent periods since an analytical distribution of missing observations

[2] To my knowledge, Working(1960) is an early paper that illustrates this example and a more comprehensive study of temporal aggregation with various statistical models is shown in Marcellino (1999).

[3] Diebolt & Robert (1994)

for each period can be derived from marginal distributions used in Kalman Filter updating step. This sampling scheme is similar to Elerian, Chib and Shephard (2001) in which missing observations are sampled from a marginal distribution conditioning on observations of two closest periods back and forth. Alternative to the data augmentation, Kalman Filter can be modified in two ways to evaluate the likelihood of proposed values for parameters under mixed frequency dataset without the data augmentation procedure. Thus, in order to demonstrate the advantage of the data augmentation over these alternatives, a Monte Carlo experiment on a medium scale New Keynesian model is presented. The second main finding of this paper is that data augmentation estimation delivers lower root mean squared errors for parameters of interest in a medium scale New Keynesian model.

Lastly, the medium scale New Keynesian model is brought to the actual data, and with the benchmark estimation method the average price duration implied by the monthly model is 5 months while that by the quarterly model is 20.7 months.

The paper proceeds as follows. Section 2 demonstrates the time aggregation bias of AR(1) process and a simple New Keynesian model to serve as a motivation. Section 3 presents mixed frequency estimation strategies after some preliminary introduction of notations. Section 4 briefly discusses a medium scale New Keynesian model following FV-GQ-RR(2010). Section 5 presents estimation results from MC experiments on the medium scale New Keynesian model across three methodologies and also estimation exercise with actual data. Section 6 concludes.

2 Temporal Aggregation Bias

This section shows the temporal aggregation bias issue with AR(1) model and with 3 equation New Keynesian model due to a frequency misspecification.

2.1 AR(1)

A monthly AR(1) process is converted into an ARMA(1,1) process when aggregated into quarterly frequency[4]. Given this conversion, this section demonstrates Monte-Carlo simulation results and also the time aggregation bias when this conversion is ignored and

[4]Derivation of a quarterly specification from aggregation of monthly AR(1) model is shown in the appendix.

estimated simply with quarterly AR(1) specification. Assume that the true model is

$$\begin{aligned} a_t &= \rho a_{t-1} + \sigma\varepsilon_t, \forall t = 1, 2, ..., T \\ \varepsilon_t &\sim iid \quad N(0, 1) \end{aligned}$$

Monthly observations are simulated from this model for $T = 300$ and with $10,000$ MC simulations. And suppose an econometrician observes the aggregated data at quarterly frequency with following aggregation scheme.

$$\tilde{a}_t = a_t + a_{t-1} + a_{t-2}$$

For each dataset of $10,000$ simulations, the econometrician can estimate quarterly AR(1), i.e. a misspecified model, or quarterly ARMA(1,1) if the true model is known to the econometrician. So

$$\tilde{a}_t = \tilde{\rho}_q \tilde{a}_{t-3} + \tilde{\sigma}_1 \tilde{\nu}_t$$

or

$$\tilde{a}_t = \rho_q \tilde{a}_{t-3} + \sigma_1 \nu_t + \sigma_2 \nu_{t-3}$$

Thus, given the exact conversion, the persistence parameter shoud be $\rho_q \equiv \rho^3$. The next table shows the estimation results of this persistence parameter. Assume $\rho = 0.9$ is the true value with the monthly model. $\tilde{\rho}_q$ is an estimate with quarterly AR(1) while $\tilde{\rho}_q$ is an estimate with quarterly ARMA(1,1). These estimates are an average of point estimates over MC simulations. And the value below is the standard deviation of those point estimates.

ρ	ρ^3	$\tilde{\rho}_q$	ρ_q
0.9	0.7290	0.8011 0.0285	0.7236 0.0469

It clearly shows upward bias on ρ_q with AR(1) specification and this is due to the misspecification. When estimated with a correct model, ARMA(1,1), it gives a value close to the truth. With this simple statistical model that has backward looking variable, the true model can be retrieved even when data is aggregated over time because the exact conversion from a monthly specification to a quarterly specification is known. However, this exact conversion will not be apparent with DSGE models where forward looking variables are present and thus aggregation of variables with expectations into coarser time interval is not clear.

2.2 Simple New Keynesian model

A parsimonious New Keynesian model with Calvo pricing feature is as follows in log-linearized form.

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + \varepsilon_{\mu,t} \\ E_t y_{t+1} &= y_t + R_t - E_t \pi_{t+1} \\ R_t &= \gamma_\Pi \pi_t + \gamma_y \left(y_t - y_{t-1} \right) + \varepsilon_{m,t} \end{aligned}$$

where

$$\begin{aligned} \varepsilon_{m,t} &= \sigma_m \eta_{m,t} \\ \varepsilon_{\mu,t} &= \rho_\mu \varepsilon_{\mu,t-1} + \sigma_\mu \eta_{\mu,t} \\ \kappa &\equiv \frac{\left(1-\beta\theta\right)\left(1-\theta\right)}{\theta} \end{aligned}$$

For simplicity, log-utility and inelastic labor supply are assumed and the monetary authority targets the interest rate following Taylor rule that responds to inflation and to growth rate of output. And the source of uncertainty is price markup shock $\varepsilon_{\mu,t}$ with AR(1) and monetary shock $\varepsilon_{m,t}$ with iid process. The monetary policy in this model is no longer neutral due to the price friction and thus causes a reaction of real activities. And it is well known that the degree of Calvo parameter, θ, determines the length of propagation of the real activities in response to the monetary policy shock. The observables are interest rate, inflation rate and quarterly growth rate of output. The reason why quarterly measure for output is used is to mimic the estimation with real data in which monthly growth rate of output is not observed but only quarterly. Suppose for now the quarterly growth rate of output is $y_t^Q - y_{t-3}^Q$. Two measurement errors are added to observables so that stochastic singularity is avoided[5]. So the observation equation of this model can be expressed as

$$\begin{bmatrix} R_t \\ \pi_t \\ y_t^Q - y_{t-3}^Q \end{bmatrix} = H' \xi_t + \begin{bmatrix} 0 \\ \sigma_{\nu_1} \nu_{1t} \\ \sigma_{\nu_2} \nu_{2t} \end{bmatrix}$$

[5] To avoid stochastic singularity, only one measurement error is needed but I added one more because on rare cases a numerical singularity could arise.

where H is derived from a solution of the model and state variable, $\xi_t \equiv [x_t, \varepsilon_{\mu,t}, \varepsilon_{m,t}]'$ with x_t being a vector of predetermined variables[6]. Calibrated parameters are shown in the next table[7],

β	θ	γ_y	γ_Π	ρ_μ	$\sigma_\mu, \sigma_m, \sigma_\nu$
0.9992	0.9	0.15	1.5	0.9	0.01

and simulated this model with $T = 100$ and with 100 MC simulations.

In order to estimate the model at quarterly frequency, aggregation schemes for each observable is necessary and they generally differ depending on whether the variable is flow or stock. Interest rate and inflation rate are growth rates of stock variables and thus it is relatively easier to aggregate compared to flow variables such as GDP. Since quarterly interest rate is a three months of compounded monthly interest rates,

$$R_t^Q = R_t + R_{t-1} + R_{t-2}$$

Similarly, quarterly inflation rate is inflation rate from three months prior to current month [8] so,

$$\pi_t^Q = \pi_t + \pi_{t-1} + \pi_{t-2}$$

And I follow NIPA convention of GDP aggregation which sums monthly nominal GDPs. So in log-linearized form, the real output would be

$$y_t^Q \equiv \frac{1}{3}\left[y_t + (y_{t-1} - \pi_t) + (y_{t-2} - \pi_t - \pi_{t-1})\right]$$

Since the original monthly model has already generated quarterly growth rate for output, observations from every last month of quarters can simply be collected to construct the quarterly dataset. Given these aggregation scheme, the quarterly observables are

$$\begin{bmatrix} R_t^Q \\ \pi_t^Q \\ y_t^Q - y_{t-3}^Q \end{bmatrix}$$

[6] Given the aggregation scheme explained in the following paragraph $x_t \equiv [y_{t-1}, y_{t-2}, y_{t-3}, y_{t-4}, y_{t-5}, \pi_{t-1}, \pi_{t-2}, \pi_{t-3}, \pi_{t-4}]'$

[7] I also checked with different degree of calibrated Calvo parameter, $\theta \in \{0.85, 0.75\}$ and find temporal aggregation biases. Also different values of Taylor rule parameters have not affected these findings with Calvo parameters.

[8] I also experimented having simply $\pi_t^Q \equiv \frac{1}{3}\left[(\pi_t + \pi_{t-1} + \pi_{t-2}) + (\pi_{t-1} + \pi_{t-2} + \pi_{t-3}) + (\pi_{t-2} + \pi_{t-3} + \pi_{t-4})\right]$ which did not affect the results.

Measurement errors are attached to all three observables reflecting the fact that this quarterly specified model might be potentially misspecified. Following the standard Bayesian technique (An & Schorfeheide(2006)), New Keynesian model which is analytically same as above is estimated based on those aggregated observables. The discount factor is calibrated by having $\tilde{\beta}_q = \beta^3$ so that the steady states of interest rates are consistent across two frequencies. The diffuse priors are set including Calvo parameters except Taylor rule parameters[9] are set to have reasonable acceptance rate and desirable convergence of MC chains.

$\tilde{\theta}_q$	$\tilde{\gamma}_y$	$\tilde{\gamma}_\Pi$	$\tilde{\rho}_\mu$	$\tilde{\sigma}_m, \tilde{\sigma}_\mu, \tilde{\sigma}_\nu$
$Unif\,(0,1)$	$N\,(0.15, 0.1)$	$N\,(1.5, 0.2)$	$Unif\,(0,1)$	$IG\,(0.02, 2)$

In addition to the standard quarterly model, a quarterly model in which price markup shock has ARMA(1,1) process and monetary policy shock has MA(1) process is also estimated. Although this does not necessarily have a theoretical justification, it is worth to examine whether additional MA terms can correct the biases following Smets & Wouters (2005). For the comparison of Calvo parameter specified at different time frequencies, an envelope calculation is needed. Because this parameter is a probability that the monopolistic competitive firm cannot reoptimize their prices within its specified decision time interval, the average price duration for those firms can be computed and further the implied probability under different decision time interval, say coarser interval, can be backed out from this price duration. For example, suppose $\theta = 0.9$ in the monthly model, then the average price duration of firms is 10 months[10] which is equivalent to $\frac{10}{3}$ quarters. So the implied probability[11] at quarterly frequency would then be $\theta_q = 0.7$. In principle, if there were no temporal aggregation bias on this parameter, the quarterly estimation results would be

[9] Under various calibration schemes, Taylor rule parameters had generally bad identifications in quarterly estimation but this anamoly did not affect Calvo estimates.

[10] average price duration $= \sum_{j=0}^{\infty} \theta^j = \frac{1}{1-\theta}$

[11] $\tilde{\theta} = 1 - \frac{1}{avg.price\ dura.}$

expected to have $\theta_q = 0.7$. The results are shown in the following table.

methods	θ	γ_y	γ_Π	ρ_μ	$log\sigma_{\mu_2}$	$log\sigma_\mu$	$log\sigma_m$
True	0.9 (0.7)	0.15	1.5	0.9 (0.729)	NA	-4.6052	-4.6052
Q-AR	0.8977 0.0173	0.0784 0.0154	1.4997 0.0221	0.6771 0.0142	NA	-2.5082 0.1540	-3.3380 0.3725
Q-ARMA	0.8642 0.0529	0.1207 0.0182	1.5273 0.0279	0.5959 0.0274	-0.2306 0.3832	-2.8447 0.2802	-3.0052 0.2671

The values in the parenthesis next to true values in the first row are the quarterly parameter values that are implied by our conversion schemes with Calvo parameter and persistence of AR(1). The second row shows substantial upward aggregation bias with respect to both Calvo parameter and persistence parameter of markup shock. Including moving average terms to the exogenous processes mitigates the bias on persistence parameter but does not eliminate the bias completely and moreover does not improve Calvo parameter estimate.

This is quite a surprising outcome relative to AR(1) example but can lead to an interesting conclusion that specifying a rational expectation model at different frequencies can produce a different estimates of certain structural parameters which cannot be resolved by simply adding MA terms. Hence, this difference is dfficult to be reconciled when rational expectation models are specified at different frequencies unless the model is estimated at its true frequency. So if a research believes in a model that is at higher frequency than he or she observes, this results show that modeling at the lower frequency due to data availability can be problematic.

3 Mixed Frequency Estimation Strategies

This section shifts the focus to an estimation strategy under a mixed frequency dataset without specifying DSGE model at a lower frequency that gives rise to a time aggregation bias. This section provides three different alternatives of mixed frequency esimtation. Although these strategies are all correct in a sense of the convergence of the markov chains and the asymptotic consistency of estimates but the data augmentation shows advantage in the efficiency of parameter estimates. The detailed algorithms are explained in this section and the efficiency performance of estimation methodologies are compared by root mean squared errors of parameter estimates of a medium scale New Keynesian model.

3.1 Preliminary Setup

Given equilibrium conditions of a model, those conditions are log-linearized around a deterministic steady state and can be summarized by state-space representation that allows Bayesian estimation framework to be implemented.

$$\begin{aligned} \xi_{t+1} &= F(\theta)\,\xi_t + v_{t+1}, v_t \sim N(0, Q(\theta)) \\ \eta_t &= H(\theta)'\,\xi_t + u_t, u_t \sim N(0, R(\theta)) \end{aligned}$$

where the first equation is the state equation and the second one is the measurement equation describing the evolution of the observables as functions of the endogenous and exogenous states. ξ_{t+1} is an $n_x \times 1$ vector of a latent state and η_t is an observed variable with $n_y \times 1$. Bayesian estimation is to maximize a posterior density function of state-space equations constructed by setting priors on parameters of a model and by the likelihood function.

$$p\left(\theta|\mathbf{y}^T\right) = \pi(\theta)\, L\left(\theta|\mathbf{y}^T\right)$$

Prediction error decompostion (Harvey(1980)) facilitates the evaluation of the likelihood function period by period using Kalman Filter that optimally estimates latent variables.

$$L\left(\theta; \mathbf{y}^T\right) = \prod_{t=1}^{T} \ell\left(\theta; y_t, \hat{x}_{t|t-1}\right)$$

Since the parameters of a DSGE model are highly nonlinear, Metropolis-Hastings algorithm is applied to numerically explore the shape of the posterior with parameter values. However, when data are available at multiple frequencies, this standard procedure of Bayesian estimation is not straightforward. For the observation variable, η_t, is incomplete and has missing observations which prevents from evaluating likelihood function with standard Kalman Filter.

An example with more notations need to be introduced to accommodate this mixed frequency dataset. As a practical purpose[12], this paper is restricting to a case where data is combined with monthly and quarterly time series and estimating monthly model. Hence,

[12] At least two of the following methods can also deal a situation where dataset is constructed by more than two frequencies in principle. However, practicality of estimating under this circumstance is questionable.

η_t can be partitioned into two variables,

$$\eta_t = \begin{bmatrix} z_t \\ w_t \end{bmatrix}$$

where w_t is a monthly observable while z_t is a quarterly observable. Then, the state space representation can be rewritten

$$\begin{aligned} \xi_{t+1} &= F\xi_t + v_{t+1} \\ \begin{bmatrix} z_t \\ w_t \end{bmatrix} &= \begin{bmatrix} H_z' \\ H_w' \end{bmatrix} \xi_t + \begin{bmatrix} u_t^z \\ u_t^w \end{bmatrix} \end{aligned}$$

and note that $t = 1, 2, 3..., T$ is a sequence of months[13]. In order to disentangle z_t into one that is observed and one missing, subsequence notations for time is necessary. Let q_i be the last month of every quarter in which data for both variables, z_t and w_t, are collected and $q_i - 1$ and $q_i - 2$ be the months in which only data for w_t are available. So when $Q \equiv \frac{T}{3}$

$$\begin{aligned} \{q_i\}_{i=1}^{Q} &\equiv \{3, 6, 9, ...T-3, T\} \\ \{q_i - 1\}_{i=1}^{Q} &\equiv \{2, 5, ..., T-1\} \\ \{q_i - 2\}_{i=1}^{Q} &\equiv \{1, 4, ..., T-2\} \end{aligned}$$

The history notations are as follows, the monthly variables will be

$$\mathbf{w}^t = \{w_\tau\}_{\tau=0}^{t}$$

while the quarterly observed variable is

$$\mathbf{z}^t = \tilde{\mathbf{z}}^{q_i} \cup \tilde{\mathbf{z}}^{q_i-1} \cup \tilde{\mathbf{z}}^{q_i-2}$$

where

$$i \in \{i : q_i \leq t\}$$

tildes denote the collection of only one months from each quarter. Given above notations,

[13] For the consistency of notations, let T be the last month of the last quarter.

the posterior density function of interest under mixed frequency dataset becomes

$$p\left(\theta|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right)$$

In principle, the mixed frequency estimation strategies will differ by how to evaluate likelihood function that constitutes this posterior density. In short, the stacking method will redefine the state space representation at quarterly frequency while the underlying model is monthly, Durbin-Koopman method will modify Kalman Filter in which the dimension of Kalman Filter gain changes consistent with the dimension of available data in each period, and the data augmentation method simulates the missing observations to fill the gap by demarginalizing the above posterior density so that it transforms into a joint density which is in terms of not only parameters but also of missing observations.

3.2 Data Augmentation Method

Data augmentation method is based on sampling from a joint posterior density that is constructed by "Rao-Blackwellization" or "demarginalization" of an original posterior density. In other words, an auxiliary variable that stands for missing observations is introduced and filling this variable with a proxy value can complete the mixed frequency dataset and thus allows to evaluate the posterior density under a complete dataset. And this proxy will be simulated at every iteration of MCMC algorithm from a tractable distribution that is derived from the model under certain parameter values. So the joint density is

$$p\left(\theta|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) = \int_Z \tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) d\boldsymbol{z}$$

$$\tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) = \ell\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right)\pi\left(\theta\right)$$

The original posterior density is a marginal density of this joint density by integrating out the auxiliary variable, missing observations. Thus, this artificial extension of a posterior density function is only for the computational device and does not invalidate the inference on the structural parameters, θ, as will be shown below. So the objective is to sample the parameters and missing observations[14] jointly but this would be feasible by separating this

[14] Whenever a hat, $\hat{}$, is labeled, it means sampled values for the variables.

joint density function into two stage samplings.

$$\begin{aligned}\left\{\hat{\mathbf{z}}^{q_Q-1}, \hat{\mathbf{z}}^{q_Q-2}\right\}^{(m)} &\sim f\left(\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \hat{\theta}^{(m)}\right) \\ \hat{\theta}^{(m+1)} &\sim p\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \left\{\hat{\mathbf{z}}^{q_Q-1}, \hat{\mathbf{z}}^{q_Q-2}\right\}^{(m)}\right)\end{aligned}$$

This is a well known strategy in statistics when the joint density function of two variables is complicated. In this scheme, one variable can be easily sampled by having other variable as a condition and vice versa. The simplest example would be a mixture model of two normally distributed random variable conditional on the other variable. The bivariate distribution of this example is hardly tractable which makes difficult to sample two variables in one step but becomes much easier with two stage Gibbs sampler algorithm. And this alternating samplings will eventually converge to the desired joint distribution of interest. This convergence is proved for more general cases by Diebolt & Robert(1990, 1994) and here it merely repeats the theorem of convergence.

Corollary 1 *The sequences* $\left\{\left(\hat{\mathbf{z}}^{q_Q-1}, \hat{\mathbf{z}}^{q_Q-2}\right)^{(m)}\right\}$ *and* $\left\{\hat{\theta}^{(m)}\right\}$ *are ergodic Markov chains with respective invariant distributions* $\int_\theta \tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) d\theta$ *and* $\int_Z \tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) d\boldsymbol{z}$. *Moreover, the convergence is uniformly geometric, i.e. there exists* $0 < \rho < 1$ *and* $C > 0$ *such that*

$$\int_\theta \left| p^m\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) - p\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right)\right| d\theta \leq C\rho^m$$

where

$$p^{(m)}\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) \equiv \int_\theta K_T\left(\theta^{(m)} | \theta^{(m-1)}\right) p^{(m-1)}\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) d\theta$$

and

$$K_T\left(\theta^{(m)} | \theta^{(m-1)}\right) \equiv \int_Z p\left(\theta^{(m)} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \left\{\hat{\mathbf{z}}^{q_Q-1}, \hat{\mathbf{z}}^{q_Q-2}\right\}^{(m-1)}\right) f\left(\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \hat{\theta}^{(m-1)}\right) d\boldsymbol{z}$$

and

$$p\left(\theta | \mathbf{w}^T, \mathbf{z}^{q_Q}\right) = \int_Z \tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) d\boldsymbol{z}$$

Given this geometric convergence and additionally a finite variance of θ, the Central Limit Theorem can be applied to ensure the asymptotic consistency of parameter estimates which is an average of $\hat{\theta}$. So

Corollary 2 *The Central Limit Theorem holds, i.e. the sample estimates are consistent*

with finite variance,

$$\bar{\theta} = \frac{1}{\sqrt{M}} \sum_{m=1}^{M} \left(\theta^{(m)} - E_p \left[\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q} \right] \right) \xrightarrow{\mathcal{L}} N(0, V)$$

where

$$V = var_p \left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q} \right) < \infty$$

3.2.1 Sampling Scheme for Data Augmentation

The data augmentation step, $\left\{ \left\{ \hat{\mathbf{z}}^{q_i - 1}, \hat{\mathbf{z}}^{q_i - 2} \right\}_{i=1}^{Q} \right\}^{(m)} \sim f \left(\left\{ \mathbf{z}^{q_i - 1}, \mathbf{z}^{q_i - 2} \right\}_{i=1}^{Q} | \mathbf{w}^T, \mathbf{z}^{q_Q}, \hat{\theta}^{(m)} \right)$, in general can be implemented in different ways. One way is to simulate the whole set of missing observations at once from a distribution implied by a model. This was shown in Chiu, Eraker, Foerster, Kim & Seoane(2008) with VAR(1) model. However, this was only feasible when the target distribution from which missing observations are drawn can be derived analytically in terms of observed data and parameters of a model. But in DSGE model's estimation in which prediction errors are estimated sequentially period by period due to existence of latent variables, simulating missing observations in one step is not feasible[15] since the target distribution of a whole set of missing observations cannot be derived in general. But similar to a conditional distribution of a state variable in Kalman Filter, a target distribution for missing observation in one period can be derived in terms of parameters and observed data. Hence, data augmentation can be done by Gibbs sampling from a target distribution sequentially period by period in the first stage of MCMC algorithm. However, since data augmentation typically involves observations not only of past but also of future, a predictive distribution from Kalman Filter cannot simply used in this context. Instead, the state space form is redefined into a companion form and this facilitates derivation of distribution of missing observations in terms of observations in adjacent periods including both past and future. To illustrate this point, the standard Kalman Filter is demonstrated first and then the target distribution of data augmentation is discussed.

[15] There are some cases when DSGE model can be transformed into VAR(2) model under certain circumstances and therefore this method could be feasible. See Ravenna(2006) for more detail with this transformation. However, this paper focuses on mixed frequency estimation of DSGE models in a general framework.

Define the prediction error variance-covariance of ξ_t and that of η_t.

$$\begin{aligned}\Sigma_{t|t-1} &\equiv E\left(\xi_t - \xi_{t|t-1}\right)\left(\xi_t - \xi_{t|t-1}\right)' \\ \Omega_{t|t-1} &\equiv E\left(\eta_t - \eta_{t|t-1}\right)\left(\eta_t - \eta_{t|t-1}\right)'\end{aligned}$$

Kalman Filter gain, K_t, minimizes $\Sigma_{t|t}$ so that $\xi_{t|t}$ is optimally estimated. So Kalman Filter is

1. Starting with given $\xi_{t|t-1}$ and $\Sigma_{t|t-1}$
2. $\Omega_{t|t-1} = H'\Sigma_{t|t-1}H + R$
3. $\eta_{t|t-1} = H'\xi_{t|t-1}$
4. $K_t = \Sigma_{t|t-1}H\left(H'\Sigma_{t|t-1}H + R\right)^{-1}$
5. $\Sigma_{t|t} = \Sigma_{t|t-1} - K_t H'\Sigma_{t|t-1}$
6. $\xi_{t|t} = \xi_{t|t-1} + K_t\left(\eta_t - \eta_{t|t-1}\right)$
7. $\Sigma_{t+1|t} = F\Sigma_{t|t}F' + Q$
8. $\xi_{t+1|t} = F\xi_{t|t}$

The probabilistic interpretation of Kalman Filter, complemetary to the minimizing variance of prediction errors of the state variables, says that the state variable, ξ_t, is updated by the conditional normal distribution with a new observation η_t. Hence,

$$N\left(\left[\begin{array}{c}\xi_t \\ \eta_t\end{array}\right]|\eta^{t-1}\right) \rightarrow N\left(\xi_t|\left[\begin{array}{c}\eta^{t-1} \\ \eta_t\end{array}\right]\right)$$

The normality is due to the assumption on normally distributed structural shocks and measurement errors. Hence, given the history of observations up to period t, $\xi_{t|t}$ is optimally estimated by a conditional mean of the latter distribution. The missing observation can be treated like ξ_t and a conditional distribution can be similarly derived but instead a proxy value is sampled from this distribution instead of the conditional mean. But since observations of periods ahead are necessary in the conditional information, a trick is necessary to derive the desired distribution.

Now, redefine the state space representation in a companion form.

$$\tilde{\eta}_{q_i} = \tilde{H}\tilde{\xi}_{q_i} + \tilde{u}_{q_i}$$

$$\tilde{\xi}_{q_i} = \tilde{F}\tilde{\xi}_{q_{i-1}} + \tilde{v}_{q_i}$$

where

$$\tilde{\eta}_{q_i} = \begin{bmatrix} z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix}, \tilde{\xi}_{q_i} = \begin{bmatrix} \xi_{q_i} \\ \xi_{q_i-1} \\ \xi_{q_i-2} \end{bmatrix}$$

This form naturally entails a joint distribution of three periods observations, so if η_{q_i} is fully observed the updating would have been

$$N\left(\begin{bmatrix} \tilde{\xi}_{q_i} \\ z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix} | \tilde{\eta}^{q_{i-1}}\right) \rightarrow N\left(\tilde{\xi}_{q_i} | \begin{bmatrix} \tilde{\eta}^{q_{i-1}} \\ z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix}\right) \tag{a}$$

But since $\tilde{\eta}_{q_i}$ is not fully observed, i.e. $\{z_{q_i-2}, z_{q_i-1}\}$ is not observed, the updating the distribution of state variables and missing observations conditioning on all observed data in a quarter q_i can be

$$N\left(\begin{bmatrix} \tilde{\xi}_{q_i} \\ z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix} | \tilde{\eta}^{q_{i-1}}\right) \rightarrow N\left(\begin{bmatrix} \tilde{\xi}_{q_i} \\ z_{q_i-2} \\ z_{q_i-1} \end{bmatrix} | \begin{bmatrix} \tilde{\eta}^{q_{i-1}} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix}\right) \tag{b}$$

Hence, missing observations can be simulated from this conditional density. This distribution will still be normal and can be derived analytically[16]. So the standard Kalman Filter would have estimated state variables as in (a) when $\tilde{\eta}_{q_i}$ is fully observed, but data augmentation by simulating this target distribution as in (b) is taken since $\{z_{q_i-2}, z_{q_i-1}\}$ is not observed. Note that $\{z_{q_i-2}, z_{q_i-1}\}$ jointly sampled every quarter and can only be sampled marginally from $\tilde{\xi}_{q_i}$ because DSGE models typically have many predetermined state variables and thus have variance singular which prevents from jointly sampling with missing observations. Thus, the distribution from which missing observations $\{z_{q_i-2}, z_{q_i-1}\}$ are drawn is the marginal normal distribution. Define this distribution as f_{q_i}

$$f_{q_i} \equiv N\left(z_{q_i-2}, z_{q_i-1} | \tilde{\eta}^{q_{i-1}}, z_{q_i}, w_{q_i-2}, w_{q_i-1}, w_{q_i}\right)$$

This sampling scheme for data augmentation is similar to Elerian, Chib and Shephard (2001) in a sense that observations adjacent to missing observations are used. w_{q_i-2}, w_{q_i-1} are contemporaneous observations coming from relationships of DSGE models between endogenous variables and $\{\tilde{\eta}^{q_{i-1}}, w_{q_i}, z_{q_i}\}$ are two periods observations adjacent to missing observations. Then, all the missing observations are drawn sequentially quarter by quarter and thus this constitutes the first stage of the sampling scheme.

$$f\left(\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \hat{\theta}^{(m)}\right) = \prod_{i=1}^{Q} f_{q_i}\left(z_{q_i-2}, z_{q_i-1} | \tilde{\eta}^{q_{i-1}}, z_{q_i}, w_{q_i-2}, w_{q_i-1}, w_{q_i}\right)$$

Obviously in principle the most efficient sampling scheme would be using the whole dataset as the conditional information, but there is a numerical issue that has to be confronted. The distribution for missing observations for each quarter can also be derived from smoothing Kalman Filter and thus incorporates more information from future observations. However, the variance of the distribution for missing observations, i.e. the analogue of variance f_{q_i}, involves inverting a covariance matrix of state variables which are normally singular due to presence of predetermined variables. Computational trick is to use a generalized inverse[17] of this matrix, but this yields numerically unstable matrix to use it as variance of the target distribution. Because of this computational obstable with using smoothing Kalman Filter which is potentially most efficient, the sampling scheme of this paper resorts to using information adjacent to missing observations which is also reasonably gaining

[16] Derivation is shown in appendix.

[17] In Matlab, "pinv.m" is used for generalized inverse.

efficiency compared to alternative estimation strategies as will be shown below[18]. The data augmenting Kalman Filter can be summarized by following.

1. Starting with given $\tilde{\xi}_{q_i|q_{i-1}}$ and $\tilde{\Sigma}_{q_i|q_{i-1}}$
2. $\tilde{\Omega}_{q_i|q_{i-1}} = \tilde{H}'\tilde{\Sigma}_{q_i|q_{i-1}}\tilde{H} + \tilde{R}$
3. Simulate $\{\hat{z}_{q_i-2}, \hat{z}_{q_i-1}\} \sim f_{q_i}\left(z_{q_i-2}, z_{q_i-1}|\tilde{\eta}^{q_{i-1}}, w_{q_i-2}, w_{q_i-1}, w_{q_i}, z_{q_i}\right)$
4. $\tilde{\eta}_{q_i|q_{i-1}} = \hat{H}'\tilde{\xi}_{q_i|q_{i-1}}$
5. $K_t = \tilde{\Sigma}_{q_i|q_{i-1}}\tilde{H}\left(\tilde{H}'\tilde{\Sigma}_{q_i|q_{i-1}}\tilde{H} + \tilde{R}\right)^{-1}$
6. $\tilde{\Sigma}_{q_i|q_i} = \tilde{\Sigma}_{q_i|q_{i-1}} - \tilde{K}_t\tilde{H}'\tilde{\Sigma}_{q_i|q_{i-1}}$
7. $\tilde{\xi}_{q_i|q_i} = \tilde{\xi}_{q_i|q_{i-1}} + \tilde{K}_t\left(\hat{\eta}_t - \tilde{\eta}_{q_i|q_{i-1}}\right)$ where $\hat{\eta}_t = \begin{bmatrix} \hat{z}_{q_i-2} \\ \hat{z}_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix}$
8. $\tilde{\Sigma}_{q_{i+1}|q_i} = \tilde{F}\tilde{\Sigma}_{q_i|q_i}\tilde{F}' + \tilde{Q}$
9. $\tilde{\xi}_{q_{i+1}|q_i} = \tilde{F}\tilde{\xi}_{q_i|q_i}$

3.2.2 Multi-Block Gibbs Sampler Algorithm

After the missing observations are sampled sequentially from distributions conditioning on parameters, sampling parameters of a model in second stage takes place conditioning on this augmented dataset. The second stage is no different than the standard Metropolis-Hasting algorithm for sampling parameters conditioning on this complete dataset. Pseudo-algorithm is summarized in the following.

Pseudo-Algorithm

1. Initialize $\theta^{(0)}$

[18] Sampling scheme for choice of conditional information can further be relaxed for a case of randomly missing observations. See Kim(2009).

2. Draw $\left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\}^{(m)} \sim f\left(\left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \hat{\theta}^{(m)}\right)$

3. Evaluate $p_1\left(\theta^{(m)} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\}^{(m)}\right)$

4. Draw θ^*

5. Evaluate $p_2\left(\theta^* | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\}^{(m)}\right)$

6. *posterior odds* $\sim Unif\,(0,1)$

7. If accept, record $\theta^{(m+1)} = \theta^*$ else $\theta^{(m+1)} = \theta^{(m)}$

8. Repeat step 2~6 for $m = 1, ..., M$

Gibbs sampling stage is in step 2. It is important to save the augmented dataset in this stage to be used in both step 3 and step 5. In short, this algorithm explores the shape of the joint density function

$$\tilde{p}\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \mathbf{z}^{q_Q}\right) = \ell\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) \pi\left(\theta\right)$$

and the likelihood function is evaluated via Kalman Filter with augmented dataset

$$\ell\left(\theta, \hat{\mathbf{z}}^{q_Q-1}, \hat{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) \equiv \prod_{t=1}^{T}\left[\ell\left(\theta, \hat{z}_t | \mathbf{w}^t, \mathbf{z}^{t-1}\right)^{1-I\left(t \in \{q_i\}_{i=1}^{Q}\right)} \ell\left(\theta | \mathbf{w}^t, z_t, \mathbf{z}^{t-1}\right)^{I\left(t \in \{q_i\}_{i=1}^{Q}\right)}\right]$$

where the missing observations are sequentially drawn from

$$\left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\}^{(m)} \sim f\left(\left\{\tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2}\right\} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}, \hat{\theta}^{(m)}\right)$$

and $I\left(t \in \{q_i\}_{i=1}^{Q}\right)$ denotes an indicator function which is one if period belongs to last month of each quarter. Finally, the samples of $\left\{\theta^{(m)}\right\}_{m=1}^{M}$ are considered as the posterior distribution of parameters, and this is the marginal density function with integrating out the missing observations so that $p\left(\theta | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) = \int_Z \ell\left(\theta, \tilde{\mathbf{z}}^{q_Q-1}, \tilde{\mathbf{z}}^{q_Q-2} | \mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) \pi\left(\theta\right) d\boldsymbol{z}$.

3.3 Stacking Method

The stacking method is simply redefining the state space representation so that the observation variable is fully observed by stacking three months of observations into a one vector.

Hence, the observables are transformed into

$$\tilde{\eta}_{q_i} \equiv \begin{bmatrix} z_{q_i} \\ w_{q_i} \\ w_{q_i-1} \\ w_{q_i-2} \end{bmatrix}$$

for $\forall \{q_i\}_{i=1}^{Q}$ so that it is always observed withouth missing observations. Accordingly, the state vector can be expressed as

$$\tilde{\xi}_{q_i} \equiv \begin{bmatrix} \xi_{q_i} \\ \xi_{q_i-1} \\ \xi_{q_i-2} \end{bmatrix}$$

Then the observation equations becomes

$$\tilde{\eta}_{q_i} = \tilde{H}\tilde{\xi}_{q_i} + \tilde{u}_{q_i}$$

where

$$\tilde{H} \equiv \begin{bmatrix} H_z & 0 & 0 \\ H_w & 0 & 0 \\ 0 & H_w & 0 \\ 0 & 0 & H_w \end{bmatrix}, \qquad \tilde{u}_{q_i} \equiv \begin{bmatrix} u_{q_i}^z \\ u_{q_i}^w \\ u_{q_i-1}^w \\ u_{q_i-2}^w \end{bmatrix}$$

and state equation is

$$\tilde{\xi}_{q_i} = \tilde{F}\tilde{\xi}_{q_{i-1}} + \tilde{v}_{q_i}$$

where

$$\tilde{F} \equiv \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix}, \quad \tilde{v}_{q_i} \equiv \begin{bmatrix} I_{n_x} & F & F^2 \\ 0 & I_{n_x} & F \\ 0 & 0 & I_{n_x} \end{bmatrix} \begin{bmatrix} v_{q_i} \\ v_{q_i-1} \\ v_{q_i-2} \end{bmatrix}$$

So the posterior density in this method effectively is evaluated by assuming

$$p\left(\theta|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) = \prod_{i=1}^{Q} \ell\left(\theta|\mathbf{w}^{q_i}, \mathbf{w}^{q_i-1}, \mathbf{w}^{q_i-2}, \tilde{\mathbf{z}}^{q_i}\right) \pi\left(\theta\right)$$

However, notice that the time interval for state space equations is quarterly which implies Kalman Filter gain for optimal estimates of state variables will be updated at quar-

terly frequency. ξ_{q_i-2} will be updated conditional on the history of observations up to $q_{i-1}\,(=q_i-3)$ which is still efficient, while ξ_{q_i-1} and ξ_{q_i} are not updated with the new observation at q_{i-2} and at q_{i-1},respectively. Thus, if a monthly model is to be estimated in which the monthly observations are heavily influenced by the latent variables of the same months, this method will suffer from losing efficiency of state variable's estimates and will potentially lead to biases of parameters' estimates of the model. Furthermore, this method in general can only be applied to the case where mixed frequency data set has consistent frequency of missing observations within the same time series, i.e. it cannot be applied to the randomly missing observation case. For example, due to possibly the less sophisticated method of data collection in earlier years of a sample which is common with emerging markets, one time series can have multiple mixed frequency observations. So if an econometrician is to estimate using this type of dataset with this method, one either has to synchronize the frequency of that particular time series by aggregating into coarser frequency or has to curtail the earlier part of the sample.

3.4 Durbin-Koopman Method

Durbin-Koopman method in this paper is an extension of an example with missing observation originally shown in Durbin & Koopman's *Time Series Analysis by State Space Methods* (2001). They showed whenever η_t is all missing for that particular period as opposed to only observing partially in the case of mixed frequency dataset, they simply estimate the state variable using the optimal estimate from the previous period which was updated up to using available observations. So $\xi_{t+1|t} = \xi_{t+1|t-1} = F\xi_{t|t-1}$ instead of $\xi_{t+1|t} = F\xi_{t|t}$. In this case Kalman Filter gain is zero in period t since there is no extra information to be exploited to estimate state variables. However, in the example in which at least some observations are partially available, Kalman Filter gain can be constructed with this available information at period t. Hence, in a standard case

$$K_t = \Sigma_{t|t-1}H\left(H'\Sigma_{t|t-1}H + R\right)^{-1}$$

and the state variable is updated with η_t by

$$\xi_{t|t} = \xi_{t|t-1} + K_t\left(\eta_t - H'\xi_{t|t-1}\right)$$

Hence when only w_t is available, K_t can be a partitioned accordingly to be consistent with mixed frequency observations so that

$$K_t = \left[\begin{array}{cc} K_t^z & K_t^w \end{array} \right]$$

then the state variable can be updated by using this submatrix K_t^w,

$$\xi_{t|t} = \xi_{t|t-1} + K_t^w \left(w_t - H_w' \xi_{t|t-1} \right)$$

and when η_t is fully observed at the last month of each quarter, Kalman Filter gain is back to the standard one with a full dimension. Hence the posterior density is evaluated with

$$p\left(\theta|\mathbf{w}^T, \tilde{\mathbf{z}}^{q_Q}\right) = \prod_{t=1}^{T} \left[\begin{array}{c} \ell\left(\theta|\mathbf{w}^t, \tilde{\mathbf{z}}^{t-1}\right)^{1-I\left(t\in\{q_i\}_{i=1}^{Q}\right)} \\ \ell\left(\theta|\mathbf{w}^t, \tilde{\mathbf{z}}^{t}\right)^{I\left(t\in\{q_i\}_{i=1}^{Q}\right)} \end{array} \right] \pi\left(\theta\right)$$

So the period likelihood is evaluated based on full observations when $t \in \{q_i\}_{i=1}^{Q}$ while it is based on only partial observations when $t \notin \{q_i\}_{i=1}^{Q}$. This method still retains the original state space representation at monthly frequency and thus updates state variables monthly. However, there is still a limitation of gaining efficiency since periods in which only w_t are available suffers lack of information from missing observations on z_t. In contrast to the stacking method, this method in principle is not restricted to monthly and quarterly frequency dataset but can also be applied to randomly missing observations within time series and also possibly the dimension of those time series observed can be time varying.

3.5 Efficiency

All of above estimation strategies are equivalent in a sense that the estimates from the markov chains of $\theta^{(m)}$ are consistent. However, in reality any estimation strategy will be influenced by the potential biases due to finite sample and thus an efficiency of estimation methods is significant from the methodological point of view. Data augmentation literature has emphasized the advantage of efficiency gain from both theoretical and empirical perspectives. Gelfand & Smith (1990) and Liu, Wong & Kong(1994) have theoretically shown the smaller variance of sampled estimates with data augmentations. As such, numerous empirical works have shown the efficiency gain of data augmentation estimates by

presenting root mean squared errors[19]. Hence, in the following section a comparison of estimation methods is based on root mean squared errors of parameters of interest in a New Keynesian model.

4 Medium Scale New Keynesian Model

This model closely follows Fernandez-Villaverde, Guerron-Quintana and Rubio-Ramirez (2010) which is similar to Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2005). I adopt this model for both Monte Carlo experiment and estimation with data since it is well known and widely studied. Following paragraphs summarize this medium scale New Keynesian model and the details can be found in technical appendix of this paper.

There is a continuum of households who consumes final good, supplies differentiated labor to labor packer in monopolistic competitive labor market, invests on capital good, saves by purchasing risk free bonds, and also has access to a complete set of Arrow securities. Calvo wage setting with partial indexation is applied in intermediate labor market. Labor packer integrates the intermediate labor supply into homogenous final labor and supply it to the intermediate good producers. While differentiated labor supply induces heterogeneity of households, the complete asset market equalizes the lagrangian multipliers of households and thus yields symmetric equilibrium conditions with respect to all household's decision variables except labor supply. The utility of household is the standard separable utility between consumption and labor hours and exogenously influenced by two preference shocks that influences the wedges in the intertemporal condition and intratemporal condition. Households also earn rental income from capital management with capacity utilization cost incurred. Another source of uncertainty is coming from the marginal efficiency of investment which creates the inverse of relative price of investment good to fluctuate over time.

Intermediate good producers use rental capital and homogenous labor to produce differentiated goods with Cobb-Douglas technology and earn profits facing monopolistic competitive market with Calvo pricing. Production technology faces total factor productivity shocks. Final good producer transforms intermediate goods into a homogenous final good to be demanded by households. Government follows Taylor rule in which risk free interest

[19] $RMSE = \sqrt{Bias\left(\hat{\theta},\theta\right)^2 + var\left(\hat{\theta}\right)}$

rate is set to respond to inflation gap and to deviation of growth rate of output from trend with its own persistence. Monetary policy shock is incorporated in Taylor rule. Aggregate demand is consumption, investment and capacity utilization cost while the aggregate supply is dictated by industry wide Cobb-Douglas production which is implicity derived from aggregating Cobb-Douglas production of intermediate good producers. And due to Calvo pricing, price dispersion across intermediate good sector creates wedge betwee these aggregate demand and aggregate supply. Same applies to the labor market due to wage dispersion. In summary, there are five exogenous processes, namely two preference shocks, investment technology shock, total factor productivity shock, and monetary shocks. And risk free interest rates, wage, inflation and consumption are used as monthly observations while output and investment series as quarterly observations. Also growth rates of wage, consumption, output and investment are used as observables which is the standard practice in this literature. Thus the observables vector[20] is

$$obs_t = \begin{bmatrix} \log R_t - \log R \\ \log \pi_t - \log \Pi \\ \Delta \log w_t \\ \Delta \log c_t \\ \left(1 - L^3\right) \log \tilde{y}_t^Q \\ \left(1 - L^3\right) \log \tilde{\imath}_t^Q \end{bmatrix}$$

Note that the quarterly aggregates for output and investment are taken into account that corresponds the available data source and this aggregation scheme follows NIPA convention as shown earlier with Simple New Keynesian model.

5 Estimation Exercise

First, the economy under this model is simulated with a set of calibrated parameters, and then estimate parameters of interest in the model across alternative estimation methods under mixed frequency data. Following subsection "Monte-Carlo Experiment" shows the results of this exercise. Second, raw data are imported from NIPA and BLS, and time series in real terms are constructed following Whelan(2002), and the model is brought to this actual dataset to be estimated which is shown in "Estimation Results". Throughout the

[20] The variables are in terms of real valued levels.

estimations in the following, I fix a small set of parameters and set priors for parameters of interest in estimation to get reasonable identification.

δ	ε	η	ϕ	Φ_2
0.025/3	10	10	0	0.001

δ is depreciation rate of capital and fixed the one third of 0.025 which is standard in quarterly model. Elasticity of substitutions for differentiated labor supply and intermediate goods are fixed to be 10. ϕ is the fixed cost parameter of production technology and Φ_2 is the parameter for capacity utilization cost function which pins down the rental rate of capital in equilibrium condition.

5.0.1 Monte-Carlo Experiment

The medium scale New Keynesian model is simulated over 40 times and with sample size of 100 each. Given from these original datasets, some of observations such as GDP and investment are deleted to construct the mixed frequency dataset. Only subset of parameters of this model is brought to estimation because the convergence properties for some parameters, mostly preference parameters, generally were not desirable for this exercise. Those parameters of choice for estimates are calibrated for the true model as follows

θ_p	χ	θ_w	χ_w	γ_R	γ_y	γ_Π	ρ_φ	ρ_d	$exp\,(\sigma_s)$
0.85	0.5	0.8	0.5	0.85	0.25	1.5	0.75	0.75	0.01

σ_s denotes standard deviations of all the exogenous shocks. Priors are set for the estimation to have a reasonable acceptance rate but as loose as possible.

θ_p	χ	θ_w	χ_w
$Unif\,(0,1)$	$Be(0.5,0.4)$	$Unif\,(0,1)$	$Be(0.5,0.4)$

γ_R	γ_y	γ_Π
$Unif\,(0,1)$	$N\,(0.25,0.1)$	$N\,(1.5,0.25)$

ρ_φ	ρ_d	$exp\,(\sigma_s)$
$N\,(0.75,0.15)$	$N\,(0.75,0.15)$	$InvGamma\,(0.01,1)$

For each estimation of one dataset, parameters are drawn 500,000 times and the pos-

terior estimates are posterior modes based on the second half of these draws, i.e. 250,000 draws. Efficiency comparison results across estimation strategies are reported below.

	$\theta_p = 0.85$		$\chi = 0.5$		$\theta_w = 0.8$		$\chi_w = 0.5$	
statistics	$\bar{\theta}_p$	$RMSE$	$\bar{\chi}$	$RMSE$	$\bar{\theta}_w$	$RMSE$	$\bar{\chi}_w$	$RMSE$
$M0$	0.8500 0.0078	0.0078	0.4843 0.0636	0.0656	0.7998 0.0094	0.0094	0.4952 0.0557	0.0559
Augment	0.8518 0.0104	0.0106	0.4763 0.0775	0.0810	0.8012 0.0101	0.0102	0.4860 0.0680	0.0695
$D-K$	0.8511 0.0126	0.0127	0.4935 0.0831	0.0833	0.7996 0.0105	0.0105	0.5050 0.0730	0.0732
Stack	0.8506 0.0118	0.0118	0.4763 0.0796	0.0831	0.8010 0.0106	0.0107	0.4898 0.0777	0.0777

Top row has the parameters of interest with true values. The third row "$M0$" is the estimation with original simulated dataset, i.e. no missing observations so that all of observables are monthly and thus estimated with the standard procedure. This will serve as a benchmark estimation for the comparison across three methodologies. "*Augment*" is the estimation with the data augmentation, "$D-K$" is Durbin-Koopman method and "*Stack*" is the stacking method. Each parameter has two statistics that are mean of point estimates and root mean squared errors of these point estimates. Numbers below the mean of point estimates are standard deviations of these point estimates. Hence, lower RMSE represents more efficient estimates compared to alternative methods. In Calvo price parameter, θ_p, and the indexation parameter, χ, show a clear advantage with the data augmentation since it brings down RMSE closer to $M0$. Calvo wage parameter, θ_w, shows only a small difference while the indexation to wage, χ_w, shows more improvement for data augmentation method. Next table shows the Taylor rule parameters.

	$\gamma_R = 0.85$		$\gamma_y = 0.25$		$\gamma_\Pi = 1.5$	
statistics	$\bar{\gamma}_R$	$RMSE$	$\bar{\gamma}_y$	$RMSE$	$\bar{\gamma}_\Pi$	$RMSE$
$M0$	0.8499 0.0067	0.0067	0.2456 0.0575	0.0577	1.5136 0.0906	0.0916
Augment	0.8491 0.0080	0.0081	0.2458 0.0509	0.0511	1.5442 0.1084	0.1170
$D-K$	0.8500 0.0085	0.0085	0.2496 0.0512	0.0512	1.5239 0.1257	0.1279
Stack	0.8505 0.0083	0.0083	0.2652 0.0501	0.0524	1.5552 0.1225	0.1344

The first two parameters, γ_R and γ_y, show some mixed evidence. RMSE for the smoothing parameter are close. γ_y seems to show no difference either but $M0$ results show higher

RMSE. This was rather one of rare parameters that show inefficiency with the benchmark estimation[21]. The efficiency ranking for γ_π is consistent with most of parameters' results and this implies data augmentation is preferrable and closer to the benchmark estimation in terms of RMSE. The rest of parameters that are of less interest are reported in the Appendix and similar conclusions can be drawn.

5.0.2 Data

US data covers from 1984:Q1 to 2010:Q2 for quarterly estimation and from 1984:M1 to 2010:M6 for monthly estimation. Interest rate is the effective Federal Funds rates, and quarterly interest rate is simply compounded over three months interest rates. In case of growth rate of wage, the average wage rate for nonfarm business sector is used for quarterly estimation. But the monthly frequency wage rate was available only for total private sector which is the major subcategory of nonfarm business sector. Since using the average wage rate for total private sector at quarterly frequency instead of nonfarm business didn't show different results and thus it can be safely deduced that the wage from different scope of a sector do not play much role in monthly frequency as well. Also, this wage rates are adjusted by the ratio between employment rate[22] for the corresponding sectors and population rate so that the wage data is consistent with what is implied by the model in which there is no unemployment.

As for GDP components, the consumption is assumed to be the sum of nondurable consumptions and services and the investment to be the sum of durable consumptions and gross domestic private investments following FV-GQ-RR(2010) and output is the sum of consumption and investment. Since those series are constructed aggregates from GDP components in NIPA tables and thus do not have corresponding aggregate real variables and price indices, I follow Whelan(2002)[23] to derive real terms and price indices of those series. And the price index for this constructed consumption series is used for price level of the model by assuming consumption good as numeraire. Inflation rate is the growth rate of

[21] 10 more parameters related to exogenous processes of the medium scale New Keynesian model are estimated. Except for two parameters of these, the efficiency with the benchmark estimation was overall better.

[22] This adjustment has been also made in Smets and Wouters (2007) and Chang, Gomes and Schorfeheide (2002). US data shows that there is higher growth rates of employment rates than the population growth and thus the raw data on the growth rate of wage has a lower trend than those of per capita GDP components.

[23] Whelan(2002) discusses how Fisher's chain-aggregated data in NIPA are computed and potential pitfalls with simply adding and substracting real series from those chain aggregates.

this CPI deflator. Since consumption and output in the model are in terms of same units, the output is normalized by this CPI deflator. Quarterly ouput series used for monthly estimation is normalized by CPI in the last month of each quarter. Due to the marginal efficiency of investment, investment good is in terms of its own unit in the model and thus investment series are deflated by it own deflator and this helps to identify this investment specific technological progress.

5.0.3 Estimation Results

Since the primary focus is to compare the temporal aggregation bias on Calvo parameters without attributing the bias to the priors, except the Calvo parameters, the priors of the rest of parameters are set equivalently for both monthly and quarterly model. Below is the standard prior specifications following closely to FV-GQ-RR(2010).

$Prior$				
B	γ	ψ	κ	α
$Be\,(0.75, 0.13)$	$N\,(1, 0.25)$	$N\,(9, 3)$	$N\,(4, 1.5)$	$N\,(0.3, 0.0125)$

χ	χ_w	γ_R	γ_y	γ_π
$Be\,(0.5, 0.142)$	$Be\,(0.5, 0.102)$	$Be\,(0.75, 0.13)$	$N\,(0.15, 0.05)$	$N\,(1.5, 0.1)$

ρ_d	ρ_φ	σ_s
$Be\,(0.5, 0.142)$	$Be\,(0.5, 0.142)$	$InvGamma\,(0.1, 2)$

The prior for Calvo parameters in each frequency is set to imply equivalent average price durations. Quarterly model's prior for Calvo parameter is set with mean ,0.5, implying 6 months price duration. Thus monthly model's Calvo parameters are set with a mode of the prior being 0.833.

θ_p,θ_w	
Q	M
$Be\,(0.5, 0.28)$	$Be\,(0.833, 0.25)$

The following table is the estimation results on Calvo parameters with two different frequencies.

methods	θ_p	*PriceDuration*	θ_w	*WageDuration*
Q	0.8549 0.0163	20.67	0.7233 0.0518	10.84
Augment	0.7984 0.0259	4.96	0.8240 0.0308	5.68

As consistent with the simulation exercises with a parsimonious New Keynesian model, the calvo parameters from quarterly model is 0.855 implying approximately 20.7 months of average price duration while the monthly model when estimated with data augmentation implies approximately 5 months price duration. The gap with Calvo wage is relatively smaller than Calvo price. Wage duration for quarterly model implies 11.8 months while 5.7 months with the monthly model.

6 Conclusions

This paper investigates the temporal aggregation issue with a New Keynesian model and finds that Calvo parameter is upward biased in the sense that the quarterly model has a stronger degree of price stickiness. Monte Carlo simulation result suggests that a frequency misspecification of a New Keynesian model generates this upward bias and the estimation with data consistently confirms this finding. This paper also examines three estimation strategies to accommodate mixed frequency dataset in DSGE model's estimations and shows methodological improvements with the data augmentation method borrowed from Bayesian statistics literature.

The results and the method provided in this paper can potentially lead to another research agenda since it can address various interesting questions in macroeconomic studies. For example, this data augmentation method can naturally conduct inferences on unobserved movements of GDP at a monthly frequency and thus potentially can tune the forecasts.

7 References

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8 Appendix

8.1 Converting monthly AR(1) into quarterly ARMA(1,1)

The true monthy model is

$$\begin{aligned} a_t &= \rho_m a_{t-1} + \sigma\varepsilon_t, \forall t = 1, 2, ..., T \\ \varepsilon_t &\sim iid \;\; N(0,1) \end{aligned}$$

Then

$$\begin{aligned} a_t &= \rho_m a_{t-1} + \sigma\varepsilon_t \\ a_{t-1} &= \rho_m a_{t-2} + \sigma\varepsilon_{t-1} \\ a_{t-2} &= \rho_m a_{t-3} + \sigma\varepsilon_{t-2} \end{aligned}$$

Define

$$\tilde{a}_t \equiv a_t + a_{t-1} + a_{t-2}$$

$$\begin{aligned} \tilde{a}_t &= \rho_m (a_{t-1} + a_{t-2} + a_{t-3}) + \sigma (\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}) \\ &= \rho_m^2 (a_{t-2} + a_{t-3} + a_{t-4}) + \rho_m\sigma (\varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}) + \sigma (\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}) \end{aligned}$$

$$\begin{aligned} &= \rho_m^3 (a_{t-3} + a_{t-4} + a_{t-5}) + \rho_m^2\sigma (\varepsilon_{t-2} + \varepsilon_{t-3} + \varepsilon_{t-4}) \\ &\quad + \rho_m\sigma (\varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}) + \sigma (\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}) \end{aligned}$$

Also define

$$\tilde{\varepsilon}_t \equiv \rho_m^2\sigma (\varepsilon_{t-2} + \varepsilon_{t-3} + \varepsilon_{t-4}) + \rho_m\sigma (\varepsilon_{t-1} + \varepsilon_{t-2} + \varepsilon_{t-3}) + \sigma (\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2})$$

Then

$$\tilde{a}_t = \rho_m^3 \tilde{a}_{t-3} + \tilde{\varepsilon}_t, \forall t = 3, 6, ..., T$$

However,

$$Cov(\tilde{\varepsilon}_t, \tilde{\varepsilon}_{t-3}) = Cov\left(\rho_m^2\sigma (\varepsilon_{t-3} + \varepsilon_{t-4}), \sigma (\varepsilon_{t-3} + \varepsilon_{t-4}) + \rho_m\sigma\varepsilon_{t-4}\right) \neq 0$$

8.2 Derivation of Distribution $f_{q_i}\left(z_{q_i-2}, z_{q_i-1} | \tilde{\eta}^{q_{i-1}}, z_{q_i}, w_{q_i-2}, w_{q_i-1}, w_{q_i}\right)$

Derivation of the distribution of $\left(\begin{array}{c} \tilde{\xi}_{q_i} \\ z_{q_i-2} \\ z_{q_i-1} \end{array} \middle| \begin{array}{c} \tilde{\eta}^{q_{i-1}} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{array} \right)$ is not trivial. This section presents the derivation from a general state space form of loglinearized DSGE model:

$$\begin{aligned} \xi_{t+1} &= F\xi_t + v_{t+1}, v_t \sim N(0, Q) \\ \eta_t &= H'\xi_t + u_t, u_t \sim N(0, R) \end{aligned}$$

Suppose η_t has some missing observations. In this case, it turns useful to partition η_t into two components

$$\eta_t \equiv \begin{bmatrix} z_t \\ w_t \end{bmatrix} = \begin{bmatrix} H_z' \\ H_w' \end{bmatrix} \xi_t + \begin{bmatrix} u_t^z \\ u_t^w \end{bmatrix}$$

Now, stacking three months of variables into one vector transforms those equations to

$$\tilde{\eta}_{q_i} = \tilde{H}\tilde{\xi}_{q_i} + \tilde{u}_{q_i}$$

$$\tilde{\xi}_{q_i} = \tilde{F}\tilde{\xi}_{q_{i-1}} + \tilde{v}_{q_i}$$

where

$$\tilde{\eta}_{q_i} \equiv \begin{bmatrix} z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix}, \tilde{u}_{q_i} \equiv \begin{bmatrix} u_{q_i-2}^z \\ u_{q_i-1}^z \\ u_{q_i}^z \\ u_{q_i-2}^w \\ u_{q_i-1}^w \\ u_{q_i}^w \end{bmatrix}$$

$$\tilde{\xi}_{q_i} \equiv \begin{bmatrix} \xi_{q_i} \\ \xi_{q_i-1} \\ \xi_{q_i-2} \end{bmatrix}, \tilde{v}_{q_i} = \begin{bmatrix} \nu_{q_i} + F\nu_{q_i-1} + F^2\nu_{q_i-2} \\ \nu_{q_i-1} + F\nu_{q_i-2} \\ \nu_{q_i-2} \end{bmatrix}$$

with

$$\begin{aligned}\tilde{v}_{q_i} &\sim N\left(0, \tilde{Q}\right) \\ \tilde{u}_{q_i} &\sim N\left(0, \tilde{R}\right)\end{aligned}$$

and

$$\tilde{H} \equiv \begin{bmatrix} 0 & 0 & H_w' \\ 0 & H_w' & 0 \\ H_w' & 0 & 0 \\ 0 & 0 & H_z' \\ 0 & H_z' & 0 \\ H_z' & 0 & 0 \end{bmatrix}$$

$$\tilde{F} \equiv \begin{bmatrix} F^3 & 0 & 0 \\ F^2 & 0 & 0 \\ F & 0 & 0 \end{bmatrix}$$

Given normality of errors, the joint distribution of states and data is normal with the following mean and variance,

$$\begin{bmatrix} \tilde{\xi}_{q_i} \\ \tilde{\eta}_{q_i} \end{bmatrix} |\tilde{\eta}^{q_{i-1}} \sim N\left(\begin{bmatrix} \tilde{\xi}_{q_i|q_{i-1}} \\ \tilde{H}'\xi_{q_i|q_{i-1}} \end{bmatrix}, \begin{bmatrix} \tilde{P}_{q_i|q_{i-1}} & \tilde{P}_{q_i|q_{i-1}}\tilde{H} \\ \tilde{H}'\tilde{P}_{q_i|q_{i-1}} & \tilde{H}'\tilde{P}_{q_i|q_{i-1}}H + \tilde{R} \end{bmatrix}\right)$$

Define

$$\tilde{H}_1 \equiv \begin{bmatrix} 0 & 0 & H_w' \\ 0 & H_w' & 0 \end{bmatrix}$$

$$\tilde{H}_2 \equiv \begin{bmatrix} H_w' & 0 & 0 \\ 0 & 0 & H_z' \\ 0 & H_z' & 0 \\ H_z' & 0 & 0 \end{bmatrix}$$

$$\tilde{R}_1 \equiv Var\left(\begin{bmatrix} u^z_{q_i-2} \\ u^z_{q_i-1} \end{bmatrix}\right)$$

$$\tilde{R}_2 \equiv Var\left(\begin{bmatrix} u^z_{q_i} \\ u^w_{q_i-2} \\ u^w_{q_i-1} \\ u^w_{q_i} \end{bmatrix}\right)$$

Rewriting with partitioned matrices

$$\begin{bmatrix} \tilde{\xi}_{q_i} \\ z_{q_i-2} \\ z_{q_i-1} \\ z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix} |\tilde{\eta}^{q_{i-1}} \sim N\left(\begin{bmatrix} \tilde{\xi}_{q_i|q_{i-1}} \\ \tilde{H}_1'\xi_{q_i|q_{i-1}} \\ \tilde{H}_2'\xi_{q_i|q_{i-1}} \end{bmatrix}, \begin{bmatrix} \tilde{P}_{q_i|q_{i-1}} & \tilde{P}_{q_i|q_{i-1}}\tilde{H}_1 & \tilde{P}_{q_i|q_{i-1}}\tilde{H}_2 \\ \tilde{H}_1'\tilde{P}_{q_i|q_{i-1}} & \tilde{H}_1'\tilde{P}_{q_i|q_{i-1}}H_1 + \tilde{R}_1 & \tilde{H}_1'\tilde{P}_{q_i|q_{i-1}}H_2 \\ \tilde{H}_2'\tilde{P}_{q_i|q_{i-1}} & \tilde{H}_2'\tilde{P}_{q_i|q_{i-1}}H_1 & \tilde{H}_2'\tilde{P}_{q_i|q_{i-1}}H_2 + \tilde{R}_2 \end{bmatrix}\right)$$

$$\equiv N\left(\begin{bmatrix} \mu_\xi \\ \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_\xi & \Sigma_{\xi,1} & \Sigma_{\xi,2} \\ \Sigma'_{\xi,1} & \Sigma_1 & \Sigma_{1,2} \\ \Sigma'_{\xi,2} & \Sigma'_{1,2} & \Sigma_2 \end{bmatrix}\right)$$

So the desired normality with updated information is the following.

$$\left(\begin{array}{c|c} \tilde{\xi}_{q_i} & \tilde{\eta}^{q_{i-1}} \\ z_{q_i-2} & z_{q_i} \\ z_{q_i-1} & w_{q_i-2} \\ & w_{q_i-1} \\ & w_{q_i} \end{array}\right) \sim N\left(\tilde{\mu}, \tilde{V}\right)$$ [24]

[24] In general, if X and Y conditional on w are jointly normal

$$\begin{bmatrix} X'|w & Y'|w \end{bmatrix}' \sim N\left(\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}\right)$$

then $X\prime|y, w$ is also jointly normally distributed with the following distribution

$$X|y, w \sim N\left(\overline{x} + \Sigma_{xy}\Sigma_{yy}^{-1}\left(y - \overline{y}\right), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)$$

$$\tilde{\mu} \equiv \begin{bmatrix} \mu_\xi \\ \mu_1 \end{bmatrix} + \begin{bmatrix} \Sigma_{\xi,2} \\ \Sigma_{1,2} \end{bmatrix} \Sigma_2^{-1} \left(\begin{bmatrix} z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix} - \mu_2 \right)$$

$$\tilde{V} \equiv \begin{bmatrix} \Sigma_\xi & \Sigma_{\xi,1} \\ \Sigma'_{\xi,1} & \Sigma_1 \end{bmatrix} - \begin{bmatrix} \Sigma_{\xi,2} \\ \Sigma_{1,2} \end{bmatrix} \Sigma_2^{-1} \begin{bmatrix} \Sigma'_{\xi,2} & \Sigma'_{1,2} \end{bmatrix}$$

Since $f_{q_i}\left(z_{q_i-2}, z_{q_i-1} | \tilde{\eta}^{q_{i-1}}, z_{q_i}, w_{q_i-2}, w_{q_i-1}, w_{q_i}\right)$ is the marginal distribution of the above normal distribution

$$f_{q_i} \sim N \left(\mu_1 + \Sigma_{1,2}\Sigma_2^{-1} \left(\begin{bmatrix} z_{q_i} \\ w_{q_i-2} \\ w_{q_i-1} \\ w_{q_i} \end{bmatrix} - \mu_2 \right), \Sigma_1 - \Sigma_{1,2}\Sigma_2^{-1}\Sigma'_{1,2} \right)$$

8.3 More Monte Carlo Results

To be updated

8.4 Medium Scale New Keynesian Model

8.4.1 Households Problem

There is a continuum of households in the economy index by i which maximizes the lifetime utility function.

$$E_0 \sum_{t=0}^{\infty} \beta^t d_t \left\{ \log\left(c_{it} - b c_{it-1}\right) + \upsilon \log\left(\frac{m_{it}}{p_t}\right) - \varphi_t \psi \frac{l_{it}^{1+\gamma}}{1+\gamma} \right\}$$

where b is the parameter that controls habit persistence, d_t, is an intertemporal preference shock and φ_t is a labor supply(intratemporal) shock :

$$\begin{aligned} \log d_t &= \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t} \text{ where } \varepsilon_{d,t} \sim N(0,1) \\ \log \varphi_t &= \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t} \text{ where } \varepsilon_{\varphi,t} \sim N(0,1) \end{aligned}$$

The i^{th} household's budget constraint is given by :

$$\begin{aligned} & c_{it} + i_{it} + \frac{m_{it}}{p_t} + \frac{B_{it+1}}{p_t} + \int q_{t+1,t} a_{it+1} d\omega_{i,t+1|t} \\ = \quad & w_{it} l_{it} + \left(r_t u_{i,t} - q_t a\left[u_{it}\right]\right) \bar{k}_{i,t} + \frac{m_{it-1}}{p_t} + R_{t-1} \frac{B_{it}}{p_t} + a_{it} + T_t + F_t \end{aligned}$$

where p_t is price level of final good, w_{jt} is the real wage, r_t is the rental price of capital, $u_{jt} > 0$ is the intensity of use of capital, $q_t a\left[u_{jt}\right]$ is the physical cost of use of capital in resource terms where

$$a\left[u\right] = \gamma_1 \left(u - 1\right) + \gamma_2 \left(u - 1\right)^2$$

Here, we assume the household has technology that transforms the final good into investment good that faces this exogenous process. Thus the investment good is

$$I_{it} = \xi_t i_{it}$$

ξ_t is an investment-specific technology shock or also its inverse is interpreted as the relative price of investment good in final good unit. Its exogenous process is

$$\xi_t = \xi_{t-1} \exp\left(\Lambda_\xi + \sigma_\xi \varepsilon_{\xi,t}\right)$$

where

$$\varepsilon_{\xi,t} \sim N\left(0, 1\right)$$

Later, I substitute with stationary variable $\mu_{\xi,t} \equiv \frac{\xi_t}{\xi_{t-1}}$ so that

$$\log \mu_{\xi,t} = \Lambda_\xi + \sigma_\xi \varepsilon_{\xi,t}$$

And this investment good is newly installed to capital stock and thus the capital stock[25] evolves with

$$\bar{k}_{it+1} = \left(1 - \delta\right) \bar{k}_{it} + \left(1 - S\left(\frac{I_{it}}{I_{it-1}}\right)\right) I_{it}$$

where

$$S\left(\frac{I_{it}}{I_{it-1}}\right) = \frac{\kappa}{2} \left(\frac{I_{it}}{I_{it-1}} - \Lambda_I\right)^2$$

[25] Here, we denote $\bar{k}_t$ as installed capital stock and k_t as capital service.

is the investment adjustment cost. For ease of notation, define

$$F\left(I_{it}, I_{it-1}\right) \equiv \left(1 - S\left(\frac{I_{it}}{I_{it-1}}\right)\right) I_{it}$$

and

$$\begin{aligned} F_{1t} &= 1 - S\left(\frac{I_{it}}{I_{it-1}}\right) - S'\left(\frac{I_{it}}{I_{it-1}}\right)\frac{I_{it}}{I_{it-1}} \\ F_{2t+1} &= S'\left(\frac{I_{it+1}}{I_{it}}\right)\left(\frac{I_{it+1}}{I_{it}}\right)^2 \end{aligned}$$

Our lagrangian problem is summarized by choosing $c_{it}, B_{it}, u_{it}, \bar{k}_{it+1}, i_{it}, I_{it}, a_{it+1,t}, w_{it}, l_{it}$ to maximize

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} d_t \left\{ \log\left(c_{it} - bc_{it-1}\right) + \upsilon \log\left(\frac{m_{it}}{p_t}\right) - \varphi_t \psi \frac{l_{it}^{1+\gamma}}{1+\gamma} \right\} \\ -\lambda_{it} \left\{ \begin{array}{c} c_{it} + i_{it} + \frac{m_{it}}{p_t} + \frac{B_{it+1}}{p_t} + \int q_{t+1,t} a_{it+1,t} d\omega_{i,t+1|t} \\ -w_{it} l_{it} - \left(r_t u_{i,t} - q_t a\left[u_{it}\right]\right)\bar{k}_{i,t} - \frac{m_{it-1}}{p_t} - R_{t-1}\frac{B_{it}}{p_t} - a_{it} - T_t - F_t \\ -q_{it}\left[\bar{k}_{it+1} - \left(1-\delta\right)\bar{k}_{it} - F\left(I_{it}, I_{it-1}\right)\right] \\ -\zeta_{it}\left[I_{it} - \xi_t i_{it}\right] \end{array} \right\} \end{array} \right]$$

And HH will determine w_{it} and l_{it} by maximizing relevant part of the lagrangian under Calvo wage setting which will be characterized separately.

Household Conditions FOCs of the above problem with respect to $c_{it}, B_{it}, u_{it}, \bar{k}_{it+1}, i_{it}, I_{it}, a_{it+1|t}$ are

$$d_t \left(c_{it} - bc_{it-1}\right)^{-1} - b\beta E_t d_{t+1} \left(c_{it+1} - bc_{it}\right)^{-1} = \lambda_{it}$$

$$\lambda_{it} = \beta E_t \lambda_{it+1} \frac{R_t}{\Pi_{t+1}}$$

$$r_t = q_t a'\left[u_{it}\right]$$

$$\lambda_{it} q_{it} = \beta E_t \left\{\lambda_{it+1}\left[\left(1-\delta\right) q_{it+1} + r_{t+1} u_{it+1} - q_{t+1} a\left(u_{it+1}\right)\right]\right\}$$

$$1 = \zeta_{it} \xi_t$$

$$\lambda_{it} \zeta_{it} = \lambda_{it} q_{it} F_{1,t} + \beta E_t \lambda_{it+1} q_{it+1} F_{2,t+1}$$

$$\lambda_{it+1} q_{t+1,t} = \lambda_{it}$$

Symmetric Equilibrium Since we consider a symmetric equilibrium due to complete asset market(the complete set of state contingent Arrow securities and perfect risk sharing) so that $c_{it} = c_t, B_{it} = B_t, \lambda_{it} = \lambda_t, u_{it} = u_t, q_{it} = q_t, \zeta_{it} = \zeta_t, i_{it} = i_t, I_{it} = I_t, \bar{k}_{it} = \bar{k}_t, a_{it+1|t} = a_{t+1|t}$. After substituting $\zeta_t = \frac{1}{\xi_t}$ and rearranging,

$$d_t \left(c_t - bc_{t-1}\right)^{-1} - b\beta E_t d_{t+1} \left(c_{t+1} - bc_t\right)^{-1} = \lambda_t$$

$$\lambda_t = \beta E_t \lambda_{t+1} \frac{R_t}{\Pi_{t+1}}$$

$$r_t = q_t a' \left[u_t\right]$$

$$\lambda_t q_t = \beta E_t \left\{\lambda_{t+1} \left[\left(1-\delta\right) q_{t+1} + r_{t+1} u_{t+1} - q_{t+1} a\left(u_{t+1}\right)\right]\right\}$$

$$1 = q_t \xi_t F_{1,t} + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \xi_{t+1} \frac{1}{\mu_{\xi,t+1}} F_{2,t+1}$$

Household labor problem Calvo wage problem for houshold

$$\max_{w_{jt}} E_t \sum_{\tau=0}^{\infty} \left(\beta\theta_w\right)^{\tau} \left\{-d_t \varphi_t \psi \frac{l_{jt+1}^{1+\gamma}}{1+\gamma} + \lambda_{jt+1} \prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} w_{jt} l_{jt+\tau}\right\}$$

subject to

$$l_{jt+\tau} = \left(\prod_{s=1}^{\tau} \frac{\Pi_{t+s-1}^{\chi_w}}{\Pi_{t+s}} \frac{w_{jt}}{w_{t+\tau}}\right)^{-\eta} l_{t+\tau}^d \quad \forall j$$

This gives the law of motion

$$f_t = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^{\eta} l_t^d + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}$$

$$f_t = \psi d_t \varphi_t \left(\frac{w_t}{w_t^*}\right)^{\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta(1+\gamma)} f_{t+1}$$

The real wage index evolves :

$$w_t^{1-\eta} = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta} w_{t-1}^{1-\eta} + \left(1-\theta_w\right) w_t^{*1-\eta}$$

which can be rewritten

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t} \right)^{1-\eta} \left(\frac{w_{t-1}}{w_t} \right)^{1-\eta} + (1-\theta_w) \left(\Pi_t^{*w} \right)^{1-\eta}$$

where

$$\Pi_t^{w^*} = \frac{w_t^*}{w_t}$$

8.4.2 Firms

Final Good Producer Final good producer produces one final good in perfectly competitive market using intermediate good with following technology.

$$y_t^d = \left(\int_0^1 y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where ε controls the elasticity of substitution between intermediated goods. And thus the intermediate good producers' markup is $\frac{\varepsilon}{\varepsilon-1}$. The problem of final good producer is

$$\max_{y_{jt}} p_t y_t^d - \int_0^1 p_{jt} y_{jt} dj$$

subject to

$$y_t^d = \left(\int_0^1 y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

gives input demand function

$$y_{jt} = \left(\frac{p_{jt}}{p_t} \right)^{-\varepsilon} y_t^d \qquad \forall j$$

where the aggregate price level is

$$p_t = \left(\int_0^1 p_{jt}^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}$$

Intermediate Good Producer Intermediate good producer's technology is

$$y_{jt} = A_t k_{jt}^{\alpha} l_{jt}^{1-\alpha} - \phi z_t$$

where k_{jt} and l_{jt} are capital services and homogenous labor and A_t follows

$$A_t = A_{t-1} \exp\left(\Lambda_A + \sigma_A \varepsilon_{A,t}\right) \quad \varepsilon_{A,t} \sim N(0,1)$$

or define $\mu_{A,t} \equiv \frac{A_t}{A_{t-1}}$

$$\log \mu_{A,t} = \Lambda_A + \sigma_A \varepsilon_{A,t}$$

also

$$z_t = A_t^{\frac{1}{1-\alpha}} \xi_t^{\frac{\alpha}{1-\alpha}}$$

or define $\mu_{z,t} \equiv \frac{z_t}{z_{t-1}}$

$$\log \frac{\mu_{z,t}}{\Lambda_z} = \frac{1}{1-\alpha} \log \frac{\mu_{A,t}}{\Lambda_A} + \frac{\alpha}{1-\alpha} \log \frac{\mu_{\xi,t}}{\Lambda_\xi}$$

and

$$\Lambda_z \equiv \Lambda_A^{\frac{1}{1-\alpha}} \Lambda_\xi^{\frac{\alpha}{1-\alpha}}$$

ϕ is fixed cost parameter and usually calibrated either to zero or to guarantee zero profits in the economy at steady state.

Firms are competitive in factor markets where they confront rents, w_t and r_t^k, from l_{jt}^d and k_{jt}^d. Thus, the firm solves the static cost minimization problem,

$$\min_{l_{jt}^d, k_{jt}} w_t l_{jt}^d + r_t k_{jt}$$

subject to the production

$$y_{jt} = A_t k_{jt}^\alpha l_{jt}^{1-\alpha} - \phi z_t$$

Assuming interior solution, FOCs are

$$\begin{aligned} w_t &= \varrho\,(1-\alpha)\, A_t k_{jt}^\alpha \left(l_{jt}^d\right)^{-\alpha} \\ r_t &= \varrho \alpha A_t k_{jt}^{1-\alpha} \left(l_{jt}^d\right)^{1-\alpha} \end{aligned}$$

where ϱ is the Lagrangian multiplier. Then we can find real marginal cost mc_t by setting

$A_t k_{jt}^{\alpha} \left(l_{jt}^d\right)^{1-\alpha} = 1$. This implies

$$\begin{aligned} 1 &= A_t K_{jt}^{\alpha} \left(l_{jt}^d\right)^{1-\alpha} = A_t \left(\frac{K_{jt}}{l_{jt}^d}\right)^{\alpha} l_{jt}^d \\ &= A_t \left(\frac{w_t}{r_t}\frac{\alpha}{1-\alpha}\right)^{\alpha} l_{jt}^d \\ l_{jt}^d &= \frac{\left(\frac{\alpha}{1-\alpha}\frac{w_t}{r_t}\right)^{-\alpha}}{A_t} \end{aligned}$$

$$\begin{aligned} mc_t &= \left(\frac{1}{1-\alpha}\right) w_t l_{jt}^d \\ &= \left(\frac{1}{1-\alpha}\right) w_t \frac{\left(\frac{\alpha}{1-\alpha}\frac{w_t}{r_t}\right)^{-\alpha}}{A_t} \\ &= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{w_t^{1-\alpha} (r_t)^{\alpha}}{A_t} \end{aligned}$$

Intermediate good producer price decision Calvo Pricing decision

$$\max_{p_{it}} E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \frac{\lambda_{t+\tau}}{\lambda_t} \left\{ \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} - mc_{t+\tau} \right) y_{it+\tau} \right\}$$

subject to

$$y_{it+\tau} = \left(\prod_{s=1}^{\tau} \Pi_{t+s-1}^{\chi} \frac{p_{it}}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}^d \quad \forall i$$

The law of motion

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

$$g_t^2 = \lambda_t \Pi_t^* y_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$

$$\varepsilon g_t^1 = (\varepsilon - 1) g_t^2$$

Price level evolves

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t} \right)^{1-\varepsilon} + (1 - \theta_p) \Pi_t^{*1-\varepsilon}$$

8.4.3 Government

Gvernment sets the nominal interest rates according to the Taylor rule :

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R} \right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\gamma_\Pi} \left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}} \right)^{\gamma_y} \right)^{1-\gamma_R} \exp(m_t)$$

through open market operations that are financed through lump-sum transfers T_t such that the deficit are equal to zero :

$$T_t = \frac{\int_0^1 m_{it} di}{p_t} - \frac{\int_0^1 m_{it-1} di}{p_t} + \frac{\int_0^1 B_{it+1} di}{p_t} - R_{t-1} \frac{\int_0^1 B_{it} di}{p_t}$$

Π represents the target levels of inflation (equal to inflation in the steady state), R steady state gross return of capital, and Λ_{y^d} the steady state gross growth rate of y_t^d. The term m_t is a random shock to monetary policy that follows $m_t = \sigma_m \varepsilon_{mt}$ where ε_{mt} is distributed according to $N(0, 1)$. Consequently, the HH aggregate budget constraint is reduced to

$$c_t + \frac{1}{\xi_t} I_t = w_t l_t + (r_t u_t - q_t a[u_t]) \bar{k}_t + \Omega_t$$

8.4.4 Aggregation

The aggregate demand is

$$y_t^d = c_t + \frac{1}{\xi_t} I_t + q_t a[u_t] \bar{k}_t$$

Calvo pricing produces price dispersion in the economy, thus

$$y_t^d \int_0^1 \left(\frac{p_{jt}}{p_t} \right)^{-\varepsilon} dj = A_t K_t^{\alpha} \left(l_t^d \right)^{1-\alpha} - \phi z_t$$

By defining $v_t^p \equiv \int_0^1 \left(\frac{p_{jt}}{p_t}\right)^{-\varepsilon} dj$, with the properties of indexation under Calvo pricing,

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\,\Pi_t^{*-\varepsilon}$$

and we have

$$y_t^d = \frac{A_t k_t^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t}{v_t^p}$$

Similarly define $v_t^w \equiv \int_0^1 \left(\frac{w_{it}}{w_t}\right)^{-\eta} di$, then

$$l_t^d = \frac{1}{v_t^w} l_t^s$$

Also

$$v_t^w = \theta_w \left(\frac{w_{t-1}}{w_t}\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w)\left(\Pi_t^{w^*}\right)^{-\eta}$$

Also in capital market

$$\int_0^1 k_{jt} dj = u_t \bar{k}_t$$

and thus

$$k_{jt} = u_t \bar{k}_t$$

Also capital stock evolves

$$\bar{k}_{t+1} = (1-\delta)\,\bar{k}_t + \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t$$

8.4.5 Equilibrium Conditions

- Intermediate good producer

$$\frac{u_t \bar{k}_t}{l_t^d} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}$$

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{w_t^{1-\alpha}\,(r_t)^{\alpha}}{A_t}$$

$$g_t^1 = \lambda_t mc_t y_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

$$g_t^2 = \lambda_t \Pi_t^* y_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$

$$\varepsilon g_t^1 = (\varepsilon - 1)\, g_t^2$$

$$1 = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p)\, \Pi_t^{*1-\varepsilon}$$

- Households

$$d_t \left(c_t - bc_{t-1}\right)^{-1} - b\beta E_t d_{t+1} \left(c_{t+1} - bc_t\right)^{-1} = \lambda_t$$

$$\lambda_t = \beta E_t \lambda_{t+1} \frac{R_t}{\Pi_{t+1}}$$

$$r_t = q_t a' \left[u_t\right]$$

$$\lambda_t q_t = \beta E_t \left\{\lambda_{t+1} \left[(1-\delta)\, q_{t+1} + r_{t+1} u_{t+1} - q_{t+1} a\left(u_{t+1}\right)\right]\right\}$$

$$1 = q_t \xi_t F_{1,t} + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \xi_{t+1} \frac{1}{\mu_{\xi,t+1}} F_{2,t+1}$$

$$f_t = \frac{\eta - 1}{\eta} \left(w_t^*\right)^{1-\eta} \lambda_t w_t^{\eta} l_t^d + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta-1} f_{t+1}$$

$$f_t = \psi d_t \varphi_t \left(\frac{w_t}{w_t^*}\right)^{\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)} \left(\frac{w_{t+1}^*}{w_t^*}\right)^{\eta(1+\gamma)} f_{t+1}$$

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta} \left(\frac{w_{t-1}}{w_t}\right)^{1-\eta} + (1-\theta_w) \left(\Pi_t^{*w}\right)^{1-\eta}$$

$$\Pi_t^{w^*} = \frac{w_t^*}{w_t}$$

- Government

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{y_t^d}{y_{t-1}^d}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp\left(m_t\right)$$

- Aggregation

$$y_t^d = c_t + \frac{1}{\xi_t} I_t + q_t a\left[u_t\right] \bar{k}_t$$

$$y_t^d = \frac{A_t k_t^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi z_t}{v_t^p}$$

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + \left(1-\theta_p\right) \Pi_t^{*-\varepsilon}$$

$$v_t^w = \theta_w \left(\frac{w_{t-1}}{w_t} \frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + \left(1-\theta_w\right) \left(\Pi_t^{w^*}\right)^{-\eta}$$

$$\bar{k}_{t+1} = \left(1-\delta\right) \bar{k}_t + \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t$$

- Exogenous Process

$$A_t = A_{t-1} \exp\left(\Lambda_A + \sigma_A \varepsilon_{A,t}\right)$$

$$\xi_t = \xi_{t-1} \exp\left(\Lambda_\xi + \sigma_\xi \varepsilon_{\xi t}\right)$$

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t}$$

$$\log \varphi_t = \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t}$$

$$m_t = \sigma_m \varepsilon_{mt}$$

- Definition for growth term

$$z_t = A_t^{\frac{1}{1-\alpha}} \xi_t^{\frac{\alpha}{1-\alpha}}$$

8.4.6 Stationary Equilibrium Conditions

Preliminaries

- Variables $\bar{k}_t, l_t^d, w_t, r_t^k, mc_t, g_t^1, g_t^2, \Pi_t, \Pi_t^*, \lambda_t, y_t^d, I_t, q_t, u_t, R_t, c_t, f_t, w_t^*, \Pi_t^{w^*}, v_t^p, v_t^w, z_t$
- Stationary variables.
  - $\tilde{c}_t = \frac{c_t}{z_t}, \tilde{w}_t = \frac{w_t}{z_t}, \tilde{w}_t^* = \frac{w_t^*}{z_t}, \tilde{\imath}_t = \frac{I_t}{z_t \xi_t}, \tilde{y}_t^d = \frac{y_t^d}{z_t}$
  - $\tilde{r}_t = r_t \xi_t, \tilde{k}_{t+1} = \frac{\bar{k}_{t+1}}{z_t \xi_t}, \tilde{q}_t = q_t \xi_t, \tilde{\lambda}_t = \lambda_t z_t,$

Stationarize I

- Intermediate good producer

$$\frac{u_t\bar{k}_t}{l_t^d z_{t-1}\xi_{t-1}}\frac{z_{t-1}\xi_{t-1}}{z_t\xi_t}=\frac{\alpha}{1-\alpha}\frac{w_t}{r_t}\frac{1}{z_t\xi_t}$$

$$mc_t=\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{\left(\frac{w_t}{z_t}\right)^{1-\alpha}(r_t\xi_t)^{\alpha}}{A_t}\frac{z_t^{1-\alpha}}{\xi_t^{\alpha}}$$

$$g_t^1=\lambda_t z_t mc_t\frac{y_t^d}{z_t}+\beta\theta_p E_t\left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{-\varepsilon}g_{t+1}^1$$
$$g_t^2=\lambda_t z_t\Pi_t^*\frac{y_t^d}{z_t}+\beta\theta_p E_t\left(\frac{\Pi_t^{\chi}}{\Pi_{t+1}}\right)^{1-\varepsilon}\left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right)g_{t+1}^2$$
$$\varepsilon g_t^1=(\varepsilon-1)\,g_t^2$$

$$1=\theta_p\left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{1-\varepsilon}+(1-\theta_p)\,\Pi_t^{*1-\varepsilon}$$

- Households

$$d_t\left(\frac{c_t}{z_t}-b\frac{c_{t-1}}{z_{t-1}}\frac{z_{t-1}}{z_t}\right)^{-1}-b\beta E_t d_{t+1}\left(\frac{c_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t}-b\frac{c_t}{z_t}\right)^{-1}=\lambda_t z_t$$

$$\lambda_t z_t=\beta E_t\lambda_{t+1}z_{t+1}\frac{z_t}{z_{t+1}}\frac{R_t}{\Pi_{t+1}}$$

$$r_t\xi_t=q_t\xi_t a'\left[u_t\right]$$

$$\lambda_t z_t q_t\xi_t=\beta E_t\left\{\lambda_{t+1}z_{t+1}\frac{z_t}{z_{t+1}}\left[\begin{array}{c}(1-\delta)\,q_{t+1}\xi_{t+1}\frac{\xi_t}{\xi_{t+1}}+r_{t+1}\xi_{t+1}\frac{\xi_t}{\xi_{t+1}}u_{t+1}\\ -q_{t+1}\xi_{t+1}\frac{\xi_t}{\xi_{t+1}}a\left(u_{t+1}\right)\end{array}\right]\right\}$$

$$1=q_t\xi_t F_{1,t}+\beta E_t\frac{\lambda_{t+1}z_{t+1}}{\lambda_t z_t}\frac{z_t}{z_{t+1}}q_{t+1}\xi_{t+1}\frac{1}{\mu_{\xi,t+1}}F_{2,t+1}$$

where

$$\begin{aligned} F_{1t} &= 1 - S\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\frac{z_t\xi_t}{z_{t-1}\xi_{t-1}}\right) - S'\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\frac{z_t\xi_t}{z_{t-1}\xi_{t-1}}\right)\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\frac{z_t\xi_t}{z_{t-1}\xi_{t-1}} \\ F_{2t+1} &= S'\left(\frac{\tilde{i}_{t+1}}{\tilde{i}_t}\frac{z_{t+1}\xi_{t+1}}{z_t\xi_t}\right)\left(\frac{\tilde{i}_{t+1}}{\tilde{i}_t}\frac{z_{t+1}\xi_{t+1}}{z_t\xi_t}\right)^2 \end{aligned}$$

$$f_t = \frac{\eta-1}{\eta}\left(\frac{w_t^*}{z_t}\right)^{1-\eta}\lambda_t z_t\left(\frac{w_t}{z_t}\right)^{\eta} l_t^d + \beta\theta_w E_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{1-\eta}\left(\frac{w_{t+1}^* z_t}{w_t^* z_{t+1}}\frac{z_{t+1}}{z_t}\right)^{\eta-1} f_{t+1}$$

$$f_t = \psi d_t\varphi_t\left(\frac{w_t z_t}{w_t^* z_t}\right)^{\eta(1+\gamma)}\left(l_t^d\right)^{1+\gamma} + \beta\theta_w E_t\left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}}\right)^{-\eta(1+\gamma)}\left(\frac{w_{t+1}^* z_t}{w_t^* z_{t+1}}\frac{z_{t+1}}{z_t}\right)^{\eta(1+\gamma)} f_{t+1}$$

$$1 = \theta_w\left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta}\left(\frac{\frac{w_{t-1}}{z_{t-1}}}{\frac{w_t}{z_t}}\frac{z_{t-1}}{z_t}\right)^{1-\eta} + (1-\theta_w)\left(\Pi_t^{*w}\right)^{1-\eta}$$

$$\Pi_t^{w^*} = \frac{w_t^*}{w_t}$$

- Government

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R}\left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi}\left(\frac{\frac{y_t^d}{y_{t-1}^d}\frac{z_{t-1}}{z_t}\frac{z_t}{z_{t-1}}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R}\exp\left(m_t\right)$$

- Aggregation

$$\frac{y_t^d}{z_t} = \frac{c_t}{z_t} + \frac{1}{z_t\xi_t}I_t + \frac{q_t\xi_t a\left[u_t\right]\bar{k}_t}{z_{t-1}\xi_{t-1}}\frac{z_{t-1}\xi_{t-1}}{z_t\xi_t}$$

$$\frac{y_t^d}{z_t} = \frac{\frac{A_t}{z_t}\left(\frac{k_t}{\xi_{t-1}z_{t-1}}\right)^{\alpha}\left(\frac{\xi_{t-1}z_{t-1}}{\xi_t z_t}\right)^{\alpha}\left(\xi_t z_t\right)^{\alpha}\left(l_t^d\right)^{1-\alpha} - \phi}{v_t^p}$$

$$v_t^p = \theta_p\left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + \left(1-\theta_p\right)\Pi_t^{*-\varepsilon}$$

$$v_t^w = \theta_w\left(\frac{w_{t-1}}{w_t}\frac{z_t}{z_{t-1}}\frac{z_{t-1}}{z_t}\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + \left(1-\theta_w\right)\left(\Pi_t^{w^*}\right)^{-\eta}$$

$$\frac{\bar{k}_{t+1}}{z_t\xi_t} = \left(1-\delta\right)\frac{\bar{k}_t}{z_{t-1}\xi_{t-1}}\frac{z_{t-1}\xi_{t-1}}{z_t\xi_t} + \left(1 - S\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\frac{z_t\xi_t}{z_{t-1}\xi_{t-1}}\right)\right)\frac{i_t}{z_t\xi_t}$$

- Exogenous Process

$$A_t = A_{t-1} \exp\left(\Lambda_A + \sigma_A \varepsilon_{A,t}\right)$$

$$\xi_t = \xi_{t-1} \exp\left(\Lambda_\xi + \sigma_\xi \varepsilon_{\xi t}\right)$$

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t}$$

$$\log \varphi_t = \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t}$$

$$m_t = \sigma_m \varepsilon_{mt}$$

- Definition for growth term

$$z_t = A_t^{\frac{1}{1-\alpha}} \xi_t^{\frac{\alpha}{1-\alpha}}$$

Stationarize II

- Intermediate good producer

$$\frac{u_t \tilde{k}_t}{l_t^d} \frac{1}{\mu_{z,t}\mu_{\xi,t}} = \frac{\alpha}{1-\alpha} \frac{\tilde{w}_t}{\tilde{r}_t}$$

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(\tilde{w}_t\right)^{1-\alpha} \left(\tilde{r}_t\right)^{\alpha}$$

$$g_t^1 = \tilde{\lambda}_t mc_t \tilde{y}_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{-\varepsilon} g_{t+1}^1$$

$$g_t^2 = \tilde{\lambda}_t \Pi_t^* \tilde{y}_t^d + \beta\theta_p E_t \left(\frac{\Pi_t^\chi}{\Pi_{t+1}}\right)^{1-\varepsilon} \left(\frac{\Pi_t^*}{\Pi_{t+1}^*}\right) g_{t+1}^2$$

$$\varepsilon g_t^1 = (\varepsilon - 1)\, g_t^2$$

$$1 = \theta_p \left(\frac{\Pi_{t-1}^\chi}{\Pi_t}\right)^{1-\varepsilon} + (1-\theta_p)\, \Pi_t^{*1-\varepsilon}$$

- Households

$$d_t \left(\tilde{c}_t - b\tilde{c}_{t-1} \frac{1}{\mu_{z,t}}\right)^{-1} - b\beta E_t d_{t+1} \left(\tilde{c}_{t+1}\mu_{z,t+1} - b\tilde{c}_t\right)^{-1} = \tilde{\lambda}_t$$

$$\tilde{\lambda}_t = \beta E_t \tilde{\lambda}_{t+1} \frac{1}{\mu_{z,t+1}} \frac{R_t}{\Pi_{t+1}}$$

$$\tilde{r}_t = \tilde{q}_t a' [u_t]$$

$$\tilde{\lambda}_t \tilde{q}_t = \beta E_t \left\{ \frac{\tilde{\lambda}_{t+1}}{\mu_{\xi,t+1}\mu_{z,t+1}} \left[\begin{array}{c} (1-\delta)\,\tilde{q}_{t+1} + \tilde{r}_{t+1} u_{t+1} \\ -\tilde{q}_{t+1} a\,(u_{t+1}) \end{array} \right] \right\}$$

$$1 = \tilde{q}_t F_{1,t} + \beta E_t \frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t \mu_{z,t+1}} \frac{\tilde{q}_{t+1}}{\mu_{\xi,t+1}} F_{2,t+1}$$

where

$$\begin{aligned} F_{1t} &= 1 - S\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\mu_{z,t}\mu_{\xi,t}\right) - S'\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\mu_{z,t}\mu_{\xi,t}\right)\frac{\tilde{i}_t}{\tilde{i}_{t-1}}\mu_{z,t}\mu_{\xi,t} \\ F_{2t+1} &= S'\left(\frac{\tilde{i}_{t+1}}{\tilde{i}_t}\mu_{z,t+1}\mu_{\xi,t+1}\right)\left(\frac{\tilde{i}_{t+1}}{\tilde{i}_t}\mu_{z,t+1}\mu_{\xi,t+1}\right)^2 \end{aligned}$$

$$f_t = \frac{\eta - 1}{\eta} \left(\tilde{w}_t^*\right)^{1-\eta} \tilde{\lambda}_t \left(\tilde{w}_t\right)^{\eta} l_t^d + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}} \frac{\tilde{w}_t^*}{\tilde{w}_{t+1}^*} \frac{1}{\mu_{z,t+1}}\right)^{1-\eta} f_{t+1}$$

$$f_t = \psi d_t \varphi_t \left(\frac{\tilde{w}_t}{\tilde{w}_t^*}\right)^{\eta(1+\gamma)} \left(l_t^d\right)^{1+\gamma} + \beta\theta_w E_t \left(\frac{\Pi_t^{\chi_w}}{\Pi_{t+1}} \frac{\tilde{w}_t^*}{\tilde{w}_{t+1}^*} \frac{1}{\mu_{z,t+1}}\right)^{-\eta(1+\gamma)} f_{t+1}$$

$$1 = \theta_w \left(\frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{1-\eta} \left(\frac{\tilde{w}_{t-1}}{\tilde{w}_t} \frac{1}{\mu_{z,t}}\right)^{1-\eta} + (1-\theta_w)\left(\Pi_t^{*w}\right)^{1-\eta}$$

$$\Pi_t^{w^*} = \frac{\tilde{w}_t^*}{\tilde{w}_t}$$

- Government

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_R} \left(\left(\frac{\Pi_t}{\Pi}\right)^{\gamma_\Pi} \left(\frac{\frac{\tilde{y}_t^d}{\tilde{y}_{t-1}^d}\mu_{z,t}}{\Lambda_{y^d}}\right)^{\gamma_y}\right)^{1-\gamma_R} \exp\left(m_t\right)$$

- Aggregation

$$\tilde{y}_t^d = \tilde{c}_t + \tilde{i}_t + \frac{\tilde{q}_t a\,[u_t]\,\tilde{k}_t}{\mu_{z,t}\mu_{\xi,t}}$$

$$\tilde{y}_t^d = \frac{\frac{\mu_{A,t}}{\mu_{z,t}} \left(u_t \tilde{k}_t\right)^{\alpha} \left(l_t^d\right)^{1-\alpha} - \phi}{v_t^p}$$

$$v_t^p = \theta_p \left(\frac{\Pi_{t-1}^{\chi}}{\Pi_t}\right)^{-\varepsilon} v_{t-1}^p + (1-\theta_p)\, \Pi_t^{*-\varepsilon}$$

$$v_t^w = \theta_w \left(\frac{\tilde{w}_{t-1}}{\tilde{w}_t \mu_{z,t}} \frac{\Pi_{t-1}^{\chi_w}}{\Pi_t}\right)^{-\eta} v_{t-1}^w + (1-\theta_w) \left(\Pi_t^{w^*}\right)^{-\eta}$$

$$\tilde{k}_{t+1} = (1-\delta) \frac{\tilde{k}_t}{\mu_{z,t}\mu_{\xi,t}} + \left(1 - S\left(\frac{\tilde{i}_t}{\tilde{i}_{t-1}} \mu_{z,t}\mu_{\xi,t}\right)\right) \tilde{i}_t$$

- Exogenous Process

$$\log \mu_{A,t} = \Lambda_A + \sigma_A \varepsilon_{A,t}$$

$$\log \mu_{\xi,t} = \Lambda_\xi + \sigma_A \varepsilon_{\xi,t}$$

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \varepsilon_{d,t}$$

$$\log \varphi_t = \rho_\varphi \log \varphi_{t-1} + \sigma_\varphi \varepsilon_{\varphi,t}$$

$$m_t = \sigma_m \varepsilon_{mt}$$

- Definition for growth term

$$\log \frac{\mu_{z,t}}{\exp(\Lambda_z)} = \frac{1}{1-\alpha} \log \frac{\mu_{A,t}}{\exp(\Lambda_A)} + \frac{\alpha}{1-\alpha} \log \frac{\mu_{\xi,t}}{\exp(\Lambda_\xi)}$$

8.4.7 Steady State

Equilibrium Conditions Let $\mu^z \equiv \Lambda_z = \Lambda_A^{\frac{1}{1-\alpha}} \Lambda_k^{\frac{\alpha}{1-\alpha}}$ where $\mu^A \equiv \Lambda_A$ and $\mu^k \equiv \Lambda_k$. Given the definitions, the mean growth rate of the economy is $\Lambda_c = \Lambda_x = \Lambda_w = \Lambda_{w^*} = \Lambda_{y^d} = \Lambda_z$. $\tilde{u} = 1$ at steady state.

- Households satisfy

$$\left(\tilde{c} - \frac{b\tilde{c}}{\Lambda_z}\right)^{-1} - b\beta \left(\tilde{c}\Lambda_z - b\tilde{c}\right)^{-1} = \tilde{\lambda}$$

$$1 = \frac{\beta}{\Lambda_z \Lambda_k} \left(\tilde{r} + 1 - \delta\right)$$

$$\beta \frac{1}{\Lambda_z} \frac{R}{\Pi} = 1$$
$$\tilde{r} = \gamma_1$$
$$1 = \tilde{q}$$

$$f = \frac{\eta - 1}{\eta} \left(\tilde{w}^*\right)^{1-\eta} \left(\Pi^{*w}\right)^{-\eta} \tilde{\lambda} \left(\tilde{w}^*\right)^{\eta} l^d + \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi\Lambda_z}\right)^{1-\eta} f$$
$$f = \psi \left(\Pi^{*w}\right)^{-\eta(1+\gamma)} \left(l^d\right)^{1+\gamma} + \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi\Lambda_z}\right)^{-\eta(1+\gamma)} f$$

- Firms that can change prices set them to satisfy (4 eqs)

$$g^1 = \tilde{\lambda} mc \tilde{y}^d + \beta\theta_p \left(\frac{\Pi^{\chi}}{\Pi}\right)^{-\varepsilon} g^1$$
$$g^2 = \tilde{\lambda} \Pi^* \tilde{y}^d + \beta\theta_p \left(\frac{\Pi^{\chi}}{\Pi}\right)^{1-\varepsilon} g^2$$
$$\varepsilon g^1 = (\varepsilon - 1) g^2$$

  where
$$\Pi^{*w} = \frac{\tilde{w}^*}{\hat{w}}$$

- They rent inputs to satisfy their static minimization problem(2 eqs)

$$\frac{\tilde{k}}{l^d} = \frac{\alpha}{\alpha - 1} \frac{\tilde{w}}{\tilde{r}} \Lambda_z \Lambda_k$$
$$mc = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \left(\tilde{w}\right)^{1-\alpha} \tilde{r}^{\alpha}$$

- The wages evolve

$$1 = \theta_w \left(\Pi^{\chi_w - 1}\right)^{1-\eta} \left(\Lambda_z\right)^{-1+\eta} + (1 - \theta_w) \left(\Pi^{*w}\right)^{1-\eta}$$

- The price level evolve

$$1 = \theta_p \left(\Pi^{\chi - 1}\right)^{1-\varepsilon} + (1 - \theta_p) \Pi^{*1-\varepsilon}$$

- Markets clear

$$\tilde{y}^d = \tilde{c} + \tilde{x}$$
$$v^p \tilde{y}^d = \frac{\Lambda_A}{\Lambda_z} \tilde{k}^\alpha \left(l^d\right)^{1-\alpha} - \phi$$

where

$$\begin{aligned} v^p &= \theta_p \left(\Pi^{\chi-1}\right)^{-\varepsilon} v^p + (1-\theta_p)\, \Pi^{*-\varepsilon} \\ v^w &= \theta_w \left(\frac{\Pi^{\chi_w - 1}}{\Lambda_z}\right)^{-\eta} v^w + (1-\theta_w)\left(\Pi^{*w}\right)^{-\eta} \end{aligned}$$

and

$$\tilde{k} = (1-\delta)\frac{\tilde{k}}{\Lambda_z \Lambda_\xi} + \tilde{\imath}$$

- Exogeous processes evolve (6 eqs)

$$\begin{aligned} d &= 1 \\ \varphi &= 1 \\ \mu^k &= \Lambda_k \\ \mu^A &= \Lambda_A \\ m &= 0 \end{aligned}$$

where

$$\Lambda_z = \Lambda_A^{\frac{1}{1-\alpha}} \Lambda_k^{\frac{\alpha}{1-\alpha}}$$

Steady State computation

- Fixed Parameters

Table 1 : Fixed Parameters

β	δ	ε	η	ϕ	γ_2
0.9992	0.025/3	10	10	0	0.001

- Estimated parameters :

$$\left\{b, \gamma, \psi, \kappa, \alpha, \theta_p, \chi_p, \theta_w, \chi_w, \Lambda_k, \Lambda_A, \gamma_R, \gamma_y, \gamma_\Pi, \Pi, \rho_d, \rho_\varphi, \sigma_K, \sigma_a, \sigma_d, \sigma_\varphi, \sigma_M\right\}$$

- Free parameters : set $u = 1$ so that $\gamma_1 = \tilde{r}$.

- Parameters related to exogenous processes : $d = 1, \varphi = 1$, $m = 0$.

- Growth terms

$$\Lambda_z = \Lambda_A^{\frac{1}{1-\alpha}} \Lambda_k^{\frac{\alpha}{1-\alpha}}$$
$$\Lambda_y = \Lambda_z$$
$$\Lambda_x = \Lambda_z \Lambda_k$$

- Interest rate

$$\tilde{r} = \frac{\Lambda_z \Lambda_k}{\beta} - 1 + \delta$$
$$\gamma_1 = \tilde{r}$$
$$R = \frac{\Pi \Lambda_z}{\beta}$$

- Prices

$$\Pi^* = \left(\frac{1 - \theta_p \Pi^{-(1-\varepsilon)(1-\chi)}}{1 - \theta_p} \right)^{\frac{1}{1-\varepsilon}}$$
$$mc = \frac{\varepsilon - 1}{\varepsilon} \frac{1 - \beta \theta_p \Pi^{(1-\chi)\varepsilon}}{1 - \beta \theta_p \Pi^{-(1-\chi)(1-\varepsilon)}} \Pi^*$$

$$v^p = \frac{1 - \theta_p}{1 - \theta_p \Pi^{(1-\chi)\varepsilon}} \Pi^{*-\varepsilon}$$

- Wages

$$\Pi^{*w} = \left(\frac{1 - \theta_w \left(\Pi \right)^{(1-\eta)(\chi_w - 1)} \left(\Lambda_z \right)^{-1+\eta}}{1 - \theta_w} \right)^{\frac{1}{1-\eta}}$$

$$\tilde{w} = (1 - \alpha) \left(mc \left(\frac{\alpha}{\tilde{r}} \right)^{\alpha} \right)^{\frac{1}{1-\alpha}}$$

$$\tilde{w}^* = \tilde{w} \Pi^{w^*}$$

$$v^w = \frac{1 - \theta_w}{1 - \theta_w \Pi^{(1-\chi_w)\eta} \left(\Lambda_z \right)^{\eta}} \left(\Pi^{*w} \right)^{-\eta}$$

- Capital/labor ratio

$$\frac{\tilde{k}}{l^d} = \frac{\alpha}{\alpha - 1} \frac{\tilde{w}}{\tilde{r}} \Lambda_z \Lambda_k$$

- Assuming $\phi = 0$ or ϕ satisfying the zero profits at steady state

$$\frac{\tilde{y}^d}{l^d} = \frac{\frac{\Lambda_A}{\Lambda_z} \left(\frac{\tilde{k}}{l^d} \right)^{\alpha} - \phi}{v^p}$$

$$\frac{\tilde{k}}{l^d} \left(1 - \frac{1 - \delta}{\Lambda_z \Lambda_\xi} \right) = \frac{\tilde{i}}{l^d}$$

$$\frac{\tilde{i}}{l^d} = \frac{\tilde{k}}{l^d} \left(\frac{\Lambda_z \Lambda_\xi - 1 + \delta}{\Lambda_z \Lambda_\xi} \right)$$

$$\frac{\tilde{c}}{l^d} = \frac{\tilde{y}^d}{l^d} - \frac{\tilde{i}}{l^d}$$

- Labor demand

$$\begin{aligned} \tilde{\lambda} l^d &= \left(\frac{\tilde{c}}{l^d} - \frac{b \tilde{c}}{\Lambda_z l^d} \right)^{-1} - b\beta \left(\frac{\tilde{c}}{l^d} \Lambda_z - b \frac{\tilde{c}}{l^d} \right)^{-1} \\ &= \left(\frac{\tilde{c}}{l^d} \right)^{-1} \left(1 - \frac{b}{\Lambda_z} \beta \right) \left(1 - \frac{b}{\Lambda_z} \right)^{-1} \\ &= \left(\frac{\tilde{c}}{l^d} \right)^{-1} \frac{(\Lambda_z - b\beta)}{(\Lambda_z - b)} \end{aligned}$$

$$f = \frac{\frac{\eta-1}{\eta}\tilde{w}^* \left(\Pi^{*w}\right)^{-\eta} \tilde{\lambda} l^d}{\left(1 - \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi\Lambda_z}\right)^{1-\eta}\right)}$$

$$l^d = \left[\frac{f\left(1 - \beta\theta_w \left(\frac{\Pi^{\chi_w}}{\Pi\Lambda_z}\right)^{-\eta(1+\gamma)}\right)}{\psi \left(\Pi^{*w}\right)^{-\eta(1+\gamma)}}\right]^{\frac{1}{1+\gamma}}$$