Specification Testing for Nonparametric Structural Models with Monotonicity in Unobservables

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Objective

Test **monotonicity in scalar unobservables**, a key identifying assumption in nonparametric structural modeling

- Fully nonseparable structural relation

$$Y = m(X, A)$$

- Endogeneity accommodated by control variables/covariates

$$X \perp A \mid Z$$

- Monotonicity null hypothesis

$$\mathbb{H}_o : \quad \forall x,\ m(x, \cdot) \ \text{ is strictly increasing}$$

Outline

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1. Introduction and Motivation

- Monotonicity has been used in labor economics, industrial organization, auctions, and elsewhere
- Studied by Matzkin (2003), Chesher (2003), Altonji and Matzkin (2005), Imbens and Newey (2009), others
- Powerful identifying assumption
  - Identifies structural function m, unobservables A, all effects
  - Allows interaction between observables and unobservables
- Weakness: may be too strong – without monotonicity
  - Structural function, unobservables, and effects not identified
  - Control variables for second stage may no longer be available
  - Unobserved heterogeneity needs more careful modeling

2. Identification Under Monotonicity

Maintained Structure

- X, Z observed finite-dimensioned random vectors
- A unobserved scalar random variable
- $X \perp A \mid Z$, where Z is not measurable-$\sigma(X)$
- $Y = m(X, A)$ Y observed, m unknown
- $G(y \mid x, z) \equiv P[Y \leq y \mid X = x, Z = z]$ is invertible in y for all (x, z)

Monotonicity

- $m(x, \cdot\ )$ is strictly increasing for all x

Identification by Conditional Quantiles

Proposition 2.3 *Suppose maintained structure and monotonicity hold. Then with normalization $a = m(x^*, a)$, for all x, a, z,*

$$\begin{aligned} m(x,a) &= G^{-1}(G(a \mid x^*, z) \mid x, z) \\ A &= G^{-1}(G(Y \mid X, z) \mid x^*, z)) \\ F_{A|X}(a \mid x) &= G_{Y|X}[\, G^{-1}(G(a \mid x^*, z) \mid x, z) \mid x] \\ F_{A|Z}(a \mid z) &= F_{A|X,Z}(a \mid x, z) = G(a \mid x^*, z) \end{aligned}$$

Derivatives of m are similarly identified

3. Estimation and Specification Testing – Heuristics

- All objects of interest are functionals of G and G^{-1}

- Estimate G and G^{-1} nonparametrically: $\hat{G}$ and $\hat{G}^{-1}$

- Take $\hat{G} = \hat{G}_{p,b}$ and $\hat{G}^{-1} = \hat{G}^{-1}_{p,b}$

  p-th order local polynomial estimators with bandwidth b

Estimating m

- Simple estimator of m

$$\hat{m}_z(x,a) = \hat{G}^{-1}(\hat{G}(a \mid x^*, z) \mid x, z))$$

- Smoothed estimator of m

$$\hat{m}_H(x,a) = \int \hat{m}_z(x,a)\ dH(z)$$

consistent for pseudo-true value

$$m_H^*(x,a) = \int G^{-1}(G(a \mid x^*, z) \mid x, z))\ dH(z)$$

- Under correct specification, $m_H^* = m$

Estimating A

- Simple estimator of A

$$\hat{A}_z = \hat{G}^{-1}(\hat{G}(Y \mid X, z) \mid x^*, z))$$

- Smoothed estimators of A

$$\begin{aligned} \hat{A}_H &= \int \hat{A}_z \, dH(z) \\ \tilde{A}_H &= \hat{m}_H^{-1}(X, Y) \equiv \inf \{a : \hat{m}_H(X, a) \geq Y\} \end{aligned}$$

consistent for pseudo-true values

$$\begin{aligned} A_H^* &= \int G^{-1}(G(Y \mid X, z) \mid x^*, z)) \, dH(z) \\ A_H^\dagger &= m_H^{*-1}(X, Y) \equiv \inf \{a : m_H^*(X, a) \geq Y\} \end{aligned}$$

- Under correct specification, $A_H^* = A_H^\dagger = A$

Specification Testing

- Basic idea: compare various estimators of A

- Let $\hat{A}_{z,i} = \hat{G}^{-1}(\hat{G}(Y_i \mid X_i, z) \mid x^*, z)), \;\; H_1 \neq H_2$,

$$\hat{A}_{1,i} = \int \hat{A}_{z,i} dH_1(z) \qquad \hat{A}_{2,i} = \int \hat{A}_{z,i} dH_2(z)$$

- Specification test statistic

$$\hat{J}_n \equiv b^{d_X} \sum_{i=1}^{n} (\hat{A}_{1,i} - \hat{A}_{2,i})^2 \pi(X_i, Y_i)$$

bandwidth $b = b_n$, weight function π

4. Estimation and Specification Testing – Asymptotics

Estimating m

Theorem 4.1 *Suppose C.1-C.6 hold. Let* $x^* \in \mathcal{X}_0$ *and* $(x, a) \in \mathcal{X}_0 \times \mathcal{A}_H$. *Then*

$$\sqrt{nb^{d_X}} \left(\hat{m}_H (x, a) - m_H^* (x, a) - B_m (x, a; x^*) \right) \xrightarrow{d} N \left(0, \sigma_m^2 (x, a; x^*) \right)$$

where $B_m (x, a; x^*)$ *and* $\sigma_m^2 (x, a; x^*)$ *are specified bias and variance terms, and*

$$\sup_{(x,a) \in \mathcal{X}_0 \times \mathcal{A}_H} \left| \hat{m}_H (x, a) - m_H^* (x, a) \right| = O_P(n^{-1/2} b^{-d_X/2} \sqrt{\log n} + b^{p+1})$$

Estimating A

Corollary 4.2 *Suppose C.1-C.6 hold, and let* $x^* \in \mathcal{X}_0$ *Then conditional on* $(X_i, Y_i) \in \mathcal{X}_0 \times \mathcal{Y}_0$,

$$\sqrt{nb^{d_X}}\left(\hat{A}_{H,i} - A^*_{H,i} - B_m\left(x^*, Y_i; X_i\right)\right) \xrightarrow{d} N\left(0, \sigma^2_m\left(x^*, Y_i; X_i\right)\right)$$

Further, for i such that $(X_i, Y_i) \in \mathcal{X}_0 \times \mathcal{Y}_0$,

$$\hat{A}_{H,i} - A^*_{H,i} = O_P(n^{-1/2} b^{-d_X/2} \times \sqrt{\log n} + b^{p+1}) \text{ uniformly in } i$$

Specification Testing – Null

Theorem 5.1 *Suppose Assumptions C.1-C.9 and C.11 hold. Then under the maintained structure and monotonicity*

$$\hat{J}_n - B_{J_n} \xrightarrow{d} N\left(0, \sigma_J^2\right)$$

where σ_J^2 *is a specified variance term, and*

$$T_n \equiv \left(\hat{J}_n - \hat{B}_{J_n}\right) / \sqrt{\hat{\sigma}_{J_n}^2} \xrightarrow{d} N\left(0, 1\right)$$

where $\hat{B}_{J_n}$ *and* $\hat{\sigma}_{J_n}^2$ *are specified consistent bias and variance estimators*

Specification Testing – Local Alternatives

Let $\gamma_n \rightarrow 0$ and let non-constant $\delta_n(X, Y)$ have

$$\mu_0 \equiv \lim_{n\rightarrow\infty} E[\ \delta_n(X, Y)^2\ \pi(X, Y)\ ] < \infty.$$

Pitman local alternatives:

$$\mathbb{H}_1(\gamma_n) : \int G_n^{-1}(G_n(y \mid x, z) \mid x^*, z))d(H_1 - H_2)(z) = \gamma_n \delta_n(x, y)$$

Theorem 5.2 *Suppose Assumptions C.1-C.9 and C.11 hold. Then under* $\mathbb{H}_1(\gamma_n)$ *with* $\gamma_n = n^{-1/2} b^{-d_X/2}$,

$$T_n \xrightarrow{d} N(\mu_0/\sigma_J, 1)$$

Specification Testing – Global Alternatives

Define

$$\mu_A = E\{[\int G^{-1}(G(Y \mid X, z) \mid x^*, z))d(H_1 - H_2)(z)]^2 \pi(X, Y)\}$$

Theorem 5.3 *Suppose Assumptions C.1-C.9 and C.11 hold. If* $\mu_A > 0$, *then for any sequence* $\lambda_n = o(\sqrt{nb^{d_X}})$

$$P(T_n > \lambda_n) \to 1$$

5. Estimation and Specification Testing – Finite Samples

- Paper contains MC for estimation of m and its derivatives
- Specification testing experiments

DGP 3: $Y_i = (0.5 + 0.1X_i^2)A_i + 2\delta_0 X_i / (0.1 + e^{A_i^2/2})$

DGP 4: $Y_i = \Phi\left((X_i + 1)A_i/4\right)(X_i + 1) - 0.5\delta_0 A_i / (1 + X_i^2)$

$\Phi(\cdot)$ is standard normal CDF
$A_i = 0.5Z_i + \eta_{1i}$
$X_i = 0.25 + Z_i - 0.25Z_i^2 + \eta_{2i}$

η_{1i}, η_{2i}, and Z_i are IID $N(0,1)$, mutually independent

$\mathbb{H}_o: \ \delta_0 = 0$ monotonicity

Implementation Details for $\hat{G}_{p,b}$ and $\hat{G}_{p,b}^{-1}$

- K = product of univariate standard normal PDFs
- $p = 1$; $b = \left(c_2 S_X n^{-1/5},\ c_2 S_Z n^{-1/5}\right)$ – undersmoothing
- H_1 is CDF for $\mathcal{U}[\xi_{\epsilon_0,Z}, \xi_{1-\epsilon_{0,Z}}]$

  H_2 is scaled beta$(3,3)$ CDF on $[\xi_{\epsilon_0,Z}, \xi_{1-\epsilon_{0,Z}}]$

  $\xi_{\epsilon_0,Z} = \epsilon_0$th sample quantile of $\{Z_i\}_{i=1}^{n}$, $\epsilon_0 = 0.05$
- $N = 30$ points for numerical integration over H_1, H_2
- $\pi\left(X_i, Y_i\right) =$
  $1\left\{\xi_{\epsilon_0,X} \le X_i \le \xi_{1-\epsilon_0,X}\right\} \times 1\left\{\xi_{\epsilon_0,Y} \le Y_i \le \xi_{1-\epsilon_0,Y}\right\}$

  $\epsilon_0 = 0.0125$
- $\hat{G}_{p,b}$ and $\hat{G}_{p,b}^{-1}$ trimmed in the tails by construction

Smoothed Local Bootstrap

1. For $i = 1, ..., n$, compute $\hat{A}_i = (\hat{A}_{1,i} + \hat{A}_{2,i})/2$

$$\hat{A}_{j,i} = \int \hat{G}_{p,b}^{-1}(\hat{G}_{p,b}(Y_i \mid X_i, z) \mid x^*, z))dH_j(z)$$

2. Draw bootstrap sample $\{Z_i^*\}_{i=1}^n$ from smoothed density

$$\tilde{f}_Z(z) = n^{-1} \sum_{i=1}^{n} \phi_{\alpha_z}(Z_i - z)$$

where $\phi_\alpha(z) = \alpha^{-1}\phi(z/\alpha)$, $\alpha_z > 0$ is bandwidth

3. For $i = 1, ..., n$, given Z_i^*, draw X_i^* and A_i^* independently from

$$\tilde{f}_{X|Z}(x|Z_i^*) = \sum_{j=1}^{n} \phi_{\alpha_x}(X_j - x)\phi_{\alpha_z}(Z_j - Z_i^*) / \sum_{l=1}^{n} \phi_{\alpha_z}(Z_l - Z_i^*)$$

$$\tilde{f}_{A|Z}(a|Z_i^*) = \sum_{j=1}^{n} \phi_{\alpha_a}(\hat{A}_j - a)\phi_{\alpha_z}(Z_j - Z_i^*) / \sum_{l=1}^{n} \phi_{\alpha_z}(Z_l - Z_i^*)$$

Smoothed Local Bootstrap (cont)

4. For $i = 1, ..., n$, compute

$$Y_i^* = \left(\hat{m}_{H_1}\left(X_i^*, A_i^*\right) + \hat{m}_{H_2}\left(X_i^*, A_i^*\right)\right)/2$$

5. Compute T_n^* with $\{(X_i^*, Y_i^*, Z_i^*)\}_{i=1}^n$ replacing $\{(X_i, Y_i, Z_i)\}_{i=1}^n$

6. Repeat B times, yielding $\left\{T_{n,j}^*\right\}_{j=1}^B$

7. Calculate bootstrap p-value: $p^* \equiv B^{-1} \sum_{j=1}^B 1\left(T_{n,j}^* \geq T_n\right)$

Bootstrap Comments and Implementation Details

- Step 3 imposes conditional independence
- Step 4 imposes monotonicity
- Implementation details:
  - $\alpha_z = S_Z n^{-1/6}$, $\alpha_x = S_X n^{-1/6}$, $\alpha_a = S_A n^{-1/6}$,
  - $n = 100, 200$
  - full bootstrap
    - $B = 100$, $\#$ MC replications $= 250$
  - warp-speed bootstrap (Giacomini, Politis, and White, 2007)
    - select $c_2 = 1.5$
    - $B = 1$, $\#$ MC replications $= 500$

Table 3: Finite sample rejection frequency for DGPs 3-4

DGP	n	δ_0	Warp-speed bootstrap			Full bootstrap		
			1%	5%	10%	1%	5%	10%
3	100	0	0.018	0.042	0.124	0.008	0.080	0.140
		1	0.320	0.388	0.420	0.316	0.404	0.456
	200	0	0.020	0.060	0.148	0.020	0.076	0.140
		1	0.392	0.442	0.474	0.408	0.464	0.508
4	100	0	0.014	0.032	0.068	0.016	0.032	0.056
		1	0.288	0.576	0.660	0.456	0.656	0.724
	200	0	0.004	0.014	0.036	0.008	0.012	0.056
		1	0.356	0.564	0.688	0.476	0.712	0.792

6. Empirical Applications

- Labor Economics: Black-White Earnings Gap: Just Ability?
- Demand: Engel Curves in a Heterogeneous Population

Engel Curves in a Heterogeneous Population

$$Y = m(X_1, X_2, A)$$

- $Y \;=\; K-$vector of budget shares
- $X_1 =$ LogExp (log total expenditure – wealth)
- $X_2 =$ nKids (# of kids – observable heterogeneity measure)
- $A \;=\;$ unobservable preference heterogeneity

Objective: Test m for monotonicity in A

Instrument for Endogenous X_1 (Imbens and Newey, 2009)

$$X_1 = \phi(S, X_2, Z),$$

- $S =$ LogWage (log of labor income as in HBAI)
- $Z =$ unobserved drivers of X_1

- $(S, X_2) \perp (A, Z)$ (exogeneity) implies

$$Z = F(X_1 \mid S, X_2)$$

$$(X_1, X_2) \perp A \mid Z$$

Data

- 1995 British Family Expenditure Survey (FES) as in Lewbel (1999)
- Two adults, married or cohabiting, one or both working, head aged $20 - 55$
- Exclude households with 3 or more kids
- $n = 1655$

Variable	Food	Catering	Alcohol	Transport	Leisure	LogExp	LogWage	nKids
Mean	0.2074	0.0805	0.0578	0.2204	0.1297	5.4215	5.8581	0.6205

Implementation Details

- Kernel: product of univariate standard normal pdfs
- Local polynomial order: $p = 1$
- Bandwidths: as in simulations
- Bootstrap replications: $B = 200$

- Estimate $Z = F(X_1 \mid S, X_2)$
  - Local quadratic regression, Silverman's rule-of-thumb bandwidth
  - Asymptotic distribution unaffected

Results

	Food	Catering	Transportation	Leisure
Value of Test Statistic	1.2895	0.7336	1.5905	1.1492
p-values	≤ 0.005	≤ 0.005	≤ 0.005	0.010

7. Conclusion

- Identification and estimation of nonparametric structural models with monotonicity in unobservables
- Based on exclusion restrictions and conditional independence
- Specification test for monotonicity in scalar unobservables
- Standard normal asymptotics
- Smoothed local bootstrap performs reasonably well
- Applications to labor economics and demand
  - Fail to reject monotonicity in Black-White earnings gap study
  - Reject monotonicity in Engel curve study